Endogenous Growth: Innovation, Credit Constraints, and Stock Price Bubbles

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Abstract

We study the potential for rational bubbles in the innovation sector to affect long term economic growth. We show that stock market prices of R&D firms could include a bubble component when credit constraints are present. Bubbles are self-sustained in equilibrium by a "liquidity" premium that originates when credit constraints are relaxed. Bubbles expand borrowing and production capacity of R&D firms, stimulate innovation and increase the growth rate. Bubbles are magnified by tighter credit constraints and scarce investment opportunities. In contrast to Hirano and Yanagawa (Restud, 2017), in our model: (i) bubbles are incorporated as part of the stock price rather than providing value to an otherwise unproductive asset; (ii) bubbles can arise at any level of financial development. Finally, we show that bubbles can create permanent reallocation effects benefiting the innovation sector over other sectors.

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\section{Introduction}

Innovation drives modern economic growth (Romer 1990, Grossman and Helpman 1991, Aghion and Howitt 1992). At the same time, innovation and technological progress often correlate with bubbles. For example, in his classic book, Shiller (2015) finds that there was rapid economic growth and widespread dissemination of technological innovations in the 1920s which led to bubbles burst later in the Great Depression. Similarly, Sorescu et al. (2018) study 51 major innovations introduced between 1825 and 2000, from steam engine train to smartphone. They detect bubbles in approximately 73\% of the innovation. A well-known instance where innovation and bubbles coincided was the so called dot-com boom. In the late 1990s, between 7,000 to 10,000 new Internet companies were founded seeking to take advantage of the new possibilities open by the internet. This was a period of rapid innovation and expanded variety of internet products (Wang 2007). At the same time, the Nasdaq Composite stock market index rose 400\%.

A second feature of innovation, or research and development (R&D), activities is that they often face credit constraints (Brown, Martinsson and Petersen 2012). Studies have found that this is particularly important for small and medium size firms (Beck and Demirguc-Kunt 2006). Credit constraints are likely due to asymmetric information and lack of collateral. Information asymmetries in turn arise from the underlying characteristics of innovation since insiders have better information about the real chances of success.

This paper develops a theory of economic growth driven by innovation, innovators facing collateral constraints, households acting as venture capitalists, and stock prices of R&D firms determining the extent of R&D activities. As we show, our model can deliver rational bubbles that are sustained in equilibrium by a "premium" that arises when collateral constraints are relaxed. Our model is able to explain why bubbles can exist as an essential part of a growing economy. We use the model to show the potential effects of bubbles on innovation and long term growth.

Our baseline follows endogenous growth models with expanding varieties first developed by Romer (1990). Romer’s model is widely used to study
issues of innovation and endogenous growth. It has a final goods sector with a representative firm, a monopolistic competitive intermediate goods sector and a R&D sector. The final goods sector and intermediate goods sector in this paper are standard. Competitive final goods producers use intermediate goods, each producer of an intermediate good is a monopolist who produces a differentiated variety which rights of production are purchased from the R&D sector.

Our major difference with standard variety models of endogenous growth is in the R&D sector which is subject to credit constraints as in Kiyotaki and Moore’s (2005) and Miao and Wang (2018). We assume there are a continuum of R&D firms, owned by households, which use both capital and labor to create new varieties that are sold to intermediate producers. R&D firms have random investment opportunities which allow them to transform output into capital which is useful for R&D production. Firms use their revenue from selling their patents and intratemporal debt from firms without investment opportunities to fund their investment. R&D firms face credit constraints which are related to their equity value (feature 2). In the event of default, lenders take over the firm but some capital is loss in the process. Thus, debts cannot be larger than the taking over value.

We show that the model exhibit multiple equilibria. In particular, there are two possible balanced growth paths (BGP), a bubbleless one and a bubbly one. Bubbles exist in R&D firms’ stock price for one of the two BGP (feature 1). In particular, there may be two components existing in stock prices. One is related to the future revenue of innovation and investment activities which determines the stock fundamental value. The other one is not related to future income directly. Even a firm without any capital can still have high stock price. This part is defined as bubble.

Our baseline model not only reflects two features in innovation in an endogenous growth model, but also provides a mechanism about how bubbles affect credit constraints and innovation. The most important result of the model is bubbles stimulate innovation when there are credit constraints. There are two effects when there are bubbles. The direct one is positive while the indirect one is negative. The direct effect is the existence of bub-
bles increases the value of collateral directly so firms can borrow more from lenders when they have investment opportunities thus the R&D industry has more investment and produce more blueprints. This is similar with crowd in effects which have been studied in other literature (Hirano and Yanagawa 2017, Miao and Wang 2018).

At the same time, since stock value is also related with capital which we define as fundamental value of stock, looseness of credit constraints reduce the demand of capital. Thus capital price decreases and the value of collateral decreases. What is more, the decrease of capital price also decreases the capital revenue from producing and selling blueprints so it is a negative effect on investment. These two negative effects offset some of the crowd in effect. However, the direct effect is the dominant one thus bubbly BGP has higher growth rate than bubbleless BGP. It is worth to mention that bubbles’ size are related with credit constraints. The tighter credit constraints are binding, the bigger the bubbles are. The effects of bubbles will also be bigger when credit constraints are tighter.

Since there are effects in different directions, bubbles can affect the inside value of the firm differently from the outside value. We find that bubbles typically increase the outside value of the firm, the value to the lenders in the event of default. Thus, lenders would like to lend more to the borrowers. But bubbles may increase or decrease the inside value of the firm, the equity value to the owners. Figure 1 explains the results. In our model, there is cost for lenders taking over the firm when borrowers default so lenders only get the firm with $\xi$ of capital. $k_1$ and $k_2$ are borrowers’ capital in two cases. $V^{bb}$ and $V^{nb}$ are bubbly and bubbleless value of the firm given capital $k$. In both cases, the stock price the lenders can get is higher with bubbles. However, it is undetermined whether bubbles increase the value ($k_2$ case) or decrease the value ($k_1$ case). This is different with Miao and Wang (2018). In their paper, bubbles invariably increase the stock price while in our model it mainly increase the value, to the lenders, in the event of default. The reason why rational bubbles exist and are sustained in one of the equilibria is due to their role relaxing borrowing constraints which provides the underlying liquidity premium. Even though the revenue from growth of the bubbles
Figure 1: The relationship between equity value and capital

is less than interest rate, people still accept the bubbles because they get additional investment revenue since bubbles increase collateral.

We also study what happens when bubbles burst. We find that investment is decreasing after the burst of bubbles due to the credit constraints so capital gradually converges to the bubbleless BGP. However, capital price acts much more rapidly. The mechanism is just the same as what we have described before.

We then extend our model to study the reallocation effects of bubbles. We show that besides the effects in the baseline model, stock price bubbles in R&D sectors also attract more labor into R&D sectors which further help the innovation and economic growth.

Our model suggests bubbles in innovation sector is good for the growth so government shall not make them burst without careful conditions. The existence of bubbles means innovation sector may face tight financial constraints. Thus, the right thing government shall do is to reduce the financial frictions and help innovation firm to get enough funds when there are bubbles in innovation sector.
In the remaining part of this section, we review related literature and discuss our contribution. In section 2, we introduce the baseline model. Section 3 is about analysis of equilibria. In section 4, we derive BGP for both bubbly economy and bubbleless economy and compare the difference. We then give mechanism how bubbles work. In section 5, we study the dynamics of bubbles. We study the dynamics around the BGP and what happens when bubbles burst. Extensions are in section 6 where we have stochastic bubbles and reallocation effects. Section 7 is conclusion.

1.1 Related literature

This paper is mostly related with literature seeking to understand the connection between growth and bubbles. Tirole (1982) finds that bubbles do not exist in standard infinite period complete market models because the existence of bubbles lead to the violation of transversality conditions. However, economists find that bubbles exist in some incomplete market models. Overlapping Generations (OLG) models attract a lot of attention among such models. Samuelson (1958) is an early study using OLG model with money. In his study, money is pure bubble helping to solve lacking of debt market. Tirole’s work (1985) is a fundamental study for bubbles in OLG models and has inspired a large literature. Among them are some papers using OLG models to study growth with bubbles (e.g., Caballero, Farhi and Hammour 2006, Martin and Ventura, 2012). They find bubbles can crowd savings away from investment but they also find bubbles may also provide additional asset and encourage investment. However, These OLG papers have some disadvantages. The market incompleteness relies on the lack of market between generations which are not the reasons lead to the existence of bubbles. They are also not suitable to do realistic quantitative explorations as Hirano and Yanagawa (2017) point out.

Besides these standard OLG models. Olivier (2000) uses a continuous-time OLG model. Olivier’s study is the one which is close to our paper. He finds that bubbles in R&D sector can benefit the growth while bubbles in other type of assets may harm the economic growth. However, Olivier’s
R&D sector is very simple and does not reflect the features of innovation. Also, bubbles are not generated by any properties of R&D sector. Bubbles in R&D sectors and other sectors are generated by demographic reasons and do not have any difference.

In recent years, there are some papers using infinitely lived agents model to study bubbles. Hirano and Yanagawa (2017) build a model with financial frictions and heterogeneous investments. However, there is no innovation sector so the technology is exogenous and we cannot know the relationship between bubbles and innovation. Secondly, they use a useless asset as bubbles. Although this kind of fiat bubbles have long tradition in literature, it does not reflect what happens in innovation. For example, dot-com bubbles happen in stock price and has no relationship to any useless asset. This useless asset also leads to strong crowd out effects as in Olivier (2000). Thirdly, their model can only be used to study bubbles in countries with intermediate level of financial frictions. Bubbles do not exist in financially underdeveloped or well-developed countries. This is clearly not true in innovation because we have already seen some bubbles in the R&D sector of United States which is one of the most financially developed countries in the world. Miao and Wang (2014) build a model using stock value as collateral and have two sectors. One sector has externality while the other one does not. In their model, bubbles can relax collateral constraints. However, it does not have innovation sector so the technology of the model is exogenous.

Our paper, however, solve all the problems we discuss before. Our model follows the work by Miao and Wang (2018). Although it is not an endogenous growth model, Miao and Wang provide a novel way to think about bubbles. They use an infinitely lived agents model with credit constraints and heterogeneous investment opportunities to show that bubbles can exist in standard infinitely lived agents model with incomplete market and not violate transeversality conditions by reducing liquidity mismatch when there are investment opportunities. Our paper has the similar setting in R&D sector and find that bubbles not only affect steady state, but also increases BGP of an endogenous growth model by stimulating innovation. To the extent of our knowledge, there is no paper studying bubbles in an endogenous model with
a well defined R&D sector before. Since bubbles in our paper exist in stock price rather than on an useless asset, our paper follows Oliver (2000) and bubbles will always help innovations rather than hurt the economic growth. But our paper shows that even bubbles may not increase stock price directly, it still stimulate innovations. The mechanism in our paper are related more with innovation rather than demographics which is a exogenous variable in innovation. Also, our paper find that although Miao and Wang’s mechanism does exist, bubbles have more complex effects than Miao and Wang’s mechanism. Besides their positive effects, there are also some negative effects to offset it. Failing to consider negative effects results in overestimate the benefit of bubbles. Thus, bubbles increase stock value which can be acquired by lenders rather than borrowers. In our extension section, we also find social resource has been reallocated to R&D sector when there are bubbles. This is a new effect which help the growth of the economy. Our model is rather robust with financial development. For underdeveloped countries, bubbles are always able to exist. Even for the most developed countries, bubbles are still able to exist if the investment opportunities do not come too often. Thus, our model can be used in study bubbles in different countries.

Besides the papers in growth and bubbles. Our paper is related with papers in different fields. First of all, this paper is related with papers studying endogenous growth with innovation. Relationship between economic growth and innovation have been studied by economists for a long time both in empirical way and theoretical way. Economists find economic data provides evidence that innovation and growth are positively related (Griliches and Lichtenberg 1984, Zachariadis 2003). There are also a large number of papers focus on studying growth and innovation in theoretical way. Romer (1990) builds a model with a R&D sector where technological innovation is in the form of expanding varieties created by labor in R&D sectors and existing knowledge. It is the R&D keeps the economy growth in long term. Grossman and Helpman (1991) and Aghion and Howitt (1992) both build model to study how R&D which improve products quality have effects on growth. Both empirical and theoretical studies find that R&D has strong effects on economic growth. Our paper extend studies in this field by introduce
credit constraints and bubbles into R&D sector.

Our paper is also related to papers studying credit constraints. The seminal work of Kiyotaki and Moore (1997) introduces collateral constraints into general equilibrium and finds collateral constraints have significant effect on the whole economy. Numerous studies follow Kiyotaki and Moore to study the effect of collateral constraints (e.g., Cordoba and Ripoll 2004, Iacoviello 2005 and Liu, Wang and Zha 2013). However, most of these studies focus on business cycles. There are few theoretical papers study innovation with credit constraints. Amable, Chatelain and Ralf (2010) is one trying to study credit constraints with R&D. They find that patents created from R&D process can be used as collateral to reduce the negative effect of collateral constraints. Our study provides a novel way to think of credit constraints and R&D.

2 The baseline model

Since our model is rather complicated, we use figure 2 to help us introduce our model before we describe it in detail. Arrows in figure 2 indicate flow of resource and goods. The representative household hold shares of firms in R&D sectors and provides labor to R&D firms. The household also get income by receiving dividends and wages. R&D firms use capital and labor to produce new patents and sell patents to intermediate goods. After that, a firm in R&D sector has investment opportunity with probability $\pi$. Those who have investment opportunities borrow from those without investment opportunity and invest but they are constrained by credit constraints. Final goods are transformed into new capital when firms invest and firms trade capital after investment stage. After buying patents from R&D firms, intermediate goods producers produce intermediate goods by using final goods. They sell intermediate goods to final goods producer who use intermediate goods to produce final goods. Besides the flow of resource, figure 2 also point out R&D firm $j$ cannot borrow more than firm’s discounted value. When there are bubbles, firm $j$’s discounted value is greater than without bubble.
2.1 Households

There is a representative household in our model who has a standard utility function

$$\sum_{t=0}^{\infty} \beta^t \ln C_t$$

where $\beta$ is the discount rate and $C_t$ is the consumption in period $t$. Household provides all its labor inelastically every period and aggregate labor supply is normalized to 1. Household trades stocks of firms in R&D sectors every period and also receive dividends from stocks it holds. Household uses wages and income from trading stocks to buy consumption and do not have any other way to save. Thus, Household faces budget constraints

$$C_t + \int (V_t^j - D_t^j) \psi_{t+1}^j d\psi_t^j = \int V_t^j \psi_t^j d\psi_t^j + W_t$$
where $W_t$ is the wage rate, $V^j_t$, $D^j_t$ are R&D firm $j$’s cum-dividend equity value and dividend and $v^j_t$ is household’s holdings of firm $j$’s shares.

Transversality conditions are

$$\lim_{T \to \infty} \beta^T \frac{V^j_T v^j_T}{C_T} = 0$$

Thus, the representative household maximizes its utility function while budget constraints and transversality conditions are satisfied.

We define the growth rate of consumption $g^c_{t+1}$

$$g^c_{t+1} = \frac{C_{t+1}}{C_t} - 1$$

and

$$\rho_{t+1} = \beta \frac{C_t}{C_{t+1}} = \beta (1 + g^c_{t+1})^{-1}$$

where $\rho_{t+1}$ is the stochastic discount factor in asset pricing literature.

### 2.2 Final goods producer

In our model, there is only one kind of final goods. Let the final goods are numeraire and all consumptions, investment and inputs are using final goods. For simplicity we assume there is only one representative firm produce final goods and it is a price taker. The final goods producer uses intermediate goods to produce and the technology is

$$Y_t = A \int_{n=1}^{N_t} (X^n_t)^\sigma \ dn, 0 < \sigma < 1$$

Here $N_t$ is total number of varieties in period $t$ and $X^n_t$ is the amount of intermediate goods $n$ the final goods producer uses. $A$ denotes the technology of final goods producer.

We use $P^n_t$ to denote the price of intermediate goods $n$. Profit maximization problem of the final goods producer is
\[
\max Y_t - \int_{n=1}^{N_t} P_t^n X_t^n \, dn
\]
subject to the production function. It is easy to solve profit maximization problem and we have the demand function for intermediate goods \(n\)
\[
P_t^n = \sigma A (X_t^n)^{\sigma - 1}
\]

### 2.3 Intermediate goods producers

Intermediate good \(X_t^n\) is produced in competitive monopolistic markets. To produce an intermediate goods \(n\), an intermediate goods producer has to pay a patent fee \(\eta_n\) to the R&D firm who creates blueprint \(n\) first. After paying the patent fee, the intermediate goods producer can produce any amount of intermediate goods at any periods. The technology of intermediate goods producer is it can transform one unit of final product to one unit of \(X_t^n\). Thus his profit is
\[
(P_t^n - 1) X_t^n
\]
Since we have already had intermediate goods \(n\)’s demand function, we can find the price intermediate goods producer of goods \(n\) set
\[
P_t^n = \frac{1}{\sigma}
\]
and the amount the producer produces
\[
X_t^n = \sigma \frac{2}{\sigma} A^{\frac{1}{\sigma}}
\]
(1)
Then the profit of producing goods \(n\) every period is
\[
\left( \frac{1 - \sigma}{\sigma} \right) \sigma \frac{2}{\sigma} A^{\frac{1}{\sigma}}
\]
. Since we know that it is competitive monopolistic markets, the discounted total profits from selling goods \(n\) must be equal to the cost of buying patent
to produce goods $n$, which means

$$
\sum_{s=t}^{\infty} \rho(s, t) \left( \frac{1 - \sigma}{\sigma} \right)^{\frac{2}{\sigma - 2}} A^{\frac{1}{\sigma - 2}} = \eta_n
$$

Here $\rho(s, t) = \prod_{v=t+1}^{s} (\rho_{v+1})$ if $s \neq t$, $\rho(s, t) = 1$ if $s = t$. Since only variables in $\eta_n$ are time variables, patents created in the same period have the same price. This result gives us

$$
\sum_{s=t}^{\infty} \rho(s, t) \left( \frac{1 - \sigma}{\sigma} \right)^{\frac{2}{\sigma - 2}} A^{\frac{1}{\sigma - 2}} = \eta_n = \eta_t \quad (2)
$$

### 2.4 R&D Sector

There are a continuum of firms $j \in [0, 1]$ in R&D sector. In every period, there are three stages. We first briefly introduce the three stages and then provide details. At the first stage, firms hire labor to create new blueprints and sell them to intermediate producers as patents. During the second stage, some firms have opportunities to invest and get new capital. They can use their own fund or loans from other firms to invest. At the third stage, firms trade capital with each other.

At the beginning of period $t$, firm $j$ in R&D sector has $K^j_t$ amount of capital it accumulated at the end of period $t - 1$. Thus capital at the first stage is given. It then hires $L^j_t$ amount of labor. Technology for firm $j$ uses both capital and labor to create new blueprints. $T^j_t$ is the amount of new blueprints created by firm $j$ in period $t$. Current technology level (current amount of blueprints $N_t$) also has effect on the innovation process. The production function of R&D firm $j$ is

$$
T^j_t = Z \left( K^j_t \right)^\alpha \left( N_t L^j_t \right)^{1-\alpha}
$$

where $Z$ is an exogenous parameter. This technology of innovation means that technology has spillover effects. Every invention benefits future invention by increasing labor productivity. This property is common setting in endogenous growth models. Capital depreciates at rate $\delta$ every period. Cap-
Ital return of producing new blueprints at period \( t \) is

\[
    r_l^j K_l^j = \max_{L_l^j} \eta_t Z \left( K_l^j \right)^\alpha \left( N_t L_l^j \right)^{1-\alpha} - W_t L_l^j
\]  \hspace{1cm} (3)

It is worth to mention that capital-labor ratio for all firms in R&D sector are same. To see this, we just solve firms’ profit maximization problem and have

\[
    \frac{W_t}{N_t} = (1 - \alpha) \eta_t Z \left( \frac{K_l^j}{N_t L_l^j} \right)^\alpha
\]  \hspace{1cm} (4)

By using this result and capital return formula above we find that

\[
    r_l^j = r_t
\]

which means every firm has same capital return rate.

After firm \( j \) sells its blueprints and get the revenue comes the second stage. Every firm has a probability of \( \pi \) to have investment opportunity and those firms have investment opportunities can transform final product into capital. The technology is 1 unit of final product at period \( t \) can be transformed into 1 unit of capital. We assume the market of capital is open after the investment thus firm \( j \) has to use the profits it sells the blueprints and external source to invest. We assume the only source of external financing for \( j \) is intratemporal loans \( E_l^j \) from other firms. Those who borrow from other firms have choice between default or not default. There is no force to ensure borrowers from defaulting so borrowers are required to provide enough collateral to secure their loans. Following Miao and Wang (2018), the value of firm is used as collateral. If the owner of borrower chooses to default and escape with the fund, lenders will take over the firm to compensate their loss. However, we assume the lender may be not familiar with the borrower’s firm. There may be a cost during the take over process and the cost is \( 1 - \xi \) of total capital. Thus the credit constraints are

\[
    E_l^j \leq \rho_{t+1} V_{l+1} \left( \xi (1 - \delta) K_l^j \right)
\]

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Here $V_t(K^j_t)$ is firm $j$’s cum-dividend equity value when there is capital $K^j_t$. The credit constraints mean that if borrowers default, the discounted value of the firm left to lenders are no less than the loans so lenders do not have any loss. For borrower $j$, it is better to pay back debt $E^j_t$ than default and lose the firm values $\rho_{t+1}V^j_{t+1}((1-\delta)K^j_t)$ so there is no default in this economy.

After investment, all firms come to the third stage at which they can buy and sell capital to each other and pay the dividends. Thus the profit of investment is

$$q_tI^j_t - I^j_t$$

where $q_t$ is the price of capital and $I^j_t$ are how many capital firm $j$ plans to create by investment.

From the setting above, we can write R&D firm $j$’s cum-dividend equity value at period $t$ by using recursive form.

$$V_t(K^j_t) = (1-\pi) \max_{K^j_{t+1}, B^j_t} \left[ D^j_t + \rho_{t+1}V_{t+1}(K^j_{t+1}) \right]$$

$$+ \pi \max_{K^j_{t+1}, I^j_t, B^j_t} \left[ D^j_{t+1} + \rho_{t+1}V_{t+1}(K^j_{t+1}) \right]$$

(5)

Here $D^j_t$ and $K^j_{t+1}$ are dividend and capital for next period when there is no investment opportunity while $D^j_{t+1}$ and $K^j_{t+1}$ are dividend and next period capital when there is investment opportunity. The cum-dividend equity value now is equal to the expected value of dividend plus discounted future cum-dividend equity value when firms make best choice of debt, investment and future capital.

Firms also face some constraints. There are budget constraints (6) and (7)

$$D^j_t + q_tK^j_{t+1} + E^j_t = r^j_tK^j_t + q_t(1-\delta)K^j_t + E^j_t$$

$$D^j_{t+1} + q_tK^j_{t+1} + E^j_t + I^j_t = r^j_tK^j_t + E^j_t + q_t(1-\delta)K^j_t + q_tI^j_t$$

(6) are budget constraints when there is no investment opportunity while (7) are budget constraints where there is investment opportunity. Investment is
constrained by available fund

\[ I_t^j \leq r_t^j K_t^j + E_t^j \]  

(8)

and debt cannot violate credit constraints

\[ E_t^j \leq \rho_{t+1} V_{t+1}^j (\xi (1 - \delta) K_t^j) \]  

(9)

. Bellman equation (5) alone with (3), (6), (7), (8) and (9) consist of R&D firm \( j \)'s dynamic programming problem.

### 2.5 Competitive Equilibrium

After we describe our model, we can define competitive equilibrium. Let \( K_t = \int_0^1 K_t^j d j \), \( I_t = \int_0^1 I_t^j d j \), \( T_t = \int_0^1 T_t^j d j \) are aggregate capital, investment, new blueprints.

**Definition 1** A competitive equilibrium is defined as allocations

\[ \{ Y_t, K_t, C_t, I_t, N_t, E_t^j, T_t, L_t^j, I_t^j, K_t^j, T_t^j, Y_t^j, \psi_t^j, X_t^j \} \]  

and prices

\[ \{ w_t, P_t^j, R_t^j, q_t, \eta_t, r_t, V_t^j \} \]  

such that household maximize its utility and firms in all three sectors maximize their profits and market clearing conditions are satisfied which are stock market is clearing \( \psi_t^j = 1 \), labor market is clearing \( \int_0^1 L_t^j d j = 1 \), debt market is clearing \( \int_0^1 E_t^j d j = 0 \), capital market is clearing \( K_{t+1} = (1 - \delta) K_t + I_t \), goods market are clearing \( C_t + \int_{n=0}^N \int_0^1 X_t^j d n + I_t = Y_t \) and the amount of patent follows \( N_{t+1} = N_t + T_t \).

### 3 Analysis of Equilibria

Similar with other endogenous growth model, many variables in our model increase to infinity. Balanced growth path (BGP) is the most important result of these models. To find the BGP, we detrend variables which are increasing with time. Let \( c_t = \frac{C_t}{N_t}, k_t = \frac{K_t}{N_t}, d_t = \frac{D_t}{N_t}, t_t = \frac{T_t}{N_t}, b_t = \frac{B_t}{N_t}, w_t = \)
\[
\frac{W_t}{N_t}, 1 + g_{t+1}^N = \frac{N_{t+1}}{N_t}.
\]
Thus
\[
\rho_{t+1} = \frac{\beta c_t}{c_{t+1} \left(1 + g_{t+1}^c\right)}
\] (10)
and capital return equation can be written as
\[
r_{t} k_t = Z \eta_t (k_t)^\alpha - w_t
\] (11)

We first consider the problem of R&D section. This problem is not a contraction mapping and may have multiple solutions.

**Proposition 2** Suppose \( q > 1 \), solution of R\&D firm \( j \)'s problem is
\[
V_t (K_j^t) = a_t K_j^t + B_t
\] (12)
where
\[
a_t = r_t + q_t (1 - \delta) + \pi \left(q_t - 1\right) \left(r_t + \rho_{t+1} a_{t+1} \xi (1 - \delta)\right)
\] (13)
\[
BB_t = [1 + \pi (q_t - 1)] \rho_{t+1} B_{t+1}
\] (14)
and
\[
q_t = \rho_{t+1} a_{t+1}
\] (15).

**Proof.** Assume solution of R&D firm \( j \)'s problem is (12). Substitute (12), (6) and (7) into (5) we have
\[
a_t K_j^t + B_t = \max_{K_{t+1}^j, K_{t+1}^j, \mu_t, B_t} r_t K_j^t + q_t (1 - \delta) K_j^t
\] (16)
\[
+ \rho_{t+1} B_{t+1} + (1 - \pi) \left[-q_t K_{t+1}^j + \rho_{t+1} a_{t+1} K_{t+1}^j\right]
\]
\[
+ \pi \left[(q_t - 1) I_t^j - q_t K_{t+1}^j + \rho_{t+1} a_{t+1} K_{t+1}^j\right]
\]
and two other constraints (8) and (9) are combined to one constraint
\[
I_t^j \leq r_t K_j^t + \rho_{t+1} a_{t+1} \xi (1 - \delta) K_j^t + \rho_{t+1} E_{t+1}
\] (17)
By taking first order derivative of $K_{j,t+1}^j$ we have (15).

Since $q > 1$, firm $j$ invests as many as it can so (17) is binding. By substituting (17) into (16) and compare the left hand side and right hand side we get (13) and (14).

(15) shows that the price of capital is equal to the value it increases. This is related with Tobin’s Q theory. Tobin’s Q theory states that if the replacement cost of capital is less than the firm’s value then the firm will increase their investment to have more capital while if the replacement cost of capital is greater than the firm’s value then the firm will not invest and decrease capital. Since (15) holds, firm $j$ is indifferent between buying and selling its existing capital. Hence $K_{j,t+1}^j$ and $K_{j,t+1}^{j,j}$ are indeterminate. We know $q_t \geq 1$ because the marginal cost of producing capital is 1. When $q > 1$, firms with investment opportunities invest as many as they can so (17) is always binding. If $q_t = 1$, however, firms are indifferent in making more investment and credit constraints do not have to bind any more. This is the reason we restrict our main analysis to $q > 1$.

The two equations (13) and (14) play key roles in our model. It shows that R&D firm $j$’s cum-dividend equity value is written as $a_t K_{j,t}^j + B_t$. The first term $a_t K_{j,t}^j$ means that capital affects the equity value while the second term $B_t$ does not relate with any goods or products. In literatures about bubbles, economists define the first term as fundamental value of a firm while the second term is viewed as bubbles. To see why $a_t K_{j,t}^j$ is the fundamental value of the firm, we rewrite (13) with (15) and get

$$q_t = \rho_{t+1} r_{t+1} + \rho_{t+1} q_{t+1} (1 - \delta) + \pi \rho_{t+1} (q_{t+1} - 1) (r_{t+1} + q_{t+1} \xi (1 - \delta)) \quad (18)$$

If we use $\varphi_{t+1} = \pi (q_{t+1} - 1) (r_{t+1} + q_{t+1} \xi (1 - \delta))$ as the expected investment revenue from next period by increasing one unit of capital, we rewrite it as

$$q_t = \rho_{t+1} (r_{t+1} + \varphi_{t+1}) + \rho_{t+1} (1 - \delta) q_{t+1} \quad (19)$$
The solution of (19) is

\[ q_t = \sum_{i=t+1}^{\infty} \rho(i, t) (1 - \delta)^{i-t-1} (r_i + \varphi_i) + \frac{\Upsilon}{\rho(t, 0) (1 - \delta)^t} \]

Here \( \Upsilon \) is a constant. \( r_i \) is the revenue of one unit of capital at period \( i \) while \( \varphi_i \) is expected investment revenue for one unit of capital. By using transversality conditions, \( \Upsilon = 0 \). Thus \( q_t = \sum_{i=t+1}^{\infty} \rho(i, t) (1 - \delta)^{i-t-1} (r_i + \varphi_i) \) is total future income if firm \( j \) buy one unit of capital in period \( t \).

\[ a_t K_t^j = \frac{1}{\rho_t} \sum_{i=t}^{\infty} \rho(i, t) (1 - \delta)^{i-t-1} (r_i + \varphi_i) K_t^j \]

reflects the total expected revenue from a firm with \( K_t^j \) capital. It is just fundamental value of a firm. It is worth to mention that since our paper uses discrete model rather than continuous time model, \( a_t \) is more complicated than Miao and Wang’s (2018) model. In their model, \( a_t = q_t \) because the return \( r_{t+1} \) and depreciation is omitted in continuous time model. The second term of equity \( B_t \), however, is not related with any fundamental future revenue directly and are viewed as bubbles by economists.

Transition of bubbles comes from (14). When there are investment opportunities, bubbles can be used as collateral to increase investment profit by \( \pi(q_t - 1)B_t \). Just as the definition given by Miao and Wang (2018), \( \pi(q_t - 1) \) is liquidity premium. Later when we discuss the balanced growth path, one can easily show that bubbles grow like this do not violate transversality conditions because the growth is smaller than one when discounted by the discount factor. The reason why people bear the loss to accept such kinds of asset is it can reduce the liquidity mismatch when firms have investment opportunities but are restricted by credit constraints. This is consistent with feature 1 of R&D that rapid technological innovations often correlate with bubbles. When there are investment opportunities, bubbles help firms face credit constraints. Thus, there are more patents created and technological innovations are faster. This effect is similar with crowd in effect in most literature about bubbles. It is deserved to mention that this is just the direct
effect of bubbles. When we compare balanced growth rate between bubbly equilibrium and bubbleless equilibrium we will find there are also indirect effects of bubbles and they may offset some of crowd in effect.

Another observation of (14) is bubbles either exist from the beginning or they never appear. As we have discussed before, the dynamic programming problem is not a contraction mapping and may have multiple solutions. We have two cases here, an equilibrium with bubbles and an equilibrium without bubble. The existence of bubbles is just a consensus of the market not relating with any fundamental of a certain firm. If all agree and believe others will accept the extra values then the bubbles exist. If they do not accept or believe others will not accept the extra values there is no bubble.

Since we are focusing on detrended variables. We detrend (14) into

$$b_t = [1 + \pi (q_t - 1)] \rho_{t+1} (1 + g_{t+1}) b_{t+1}$$ (20)

Before we move on to study two different equilibria, we first discuss a little further to the general case.

We have already known every inventor has the same capital labor ratio. Thus

$$T_t = \int_0^1 T_t^j \, dj = Z (K_t)^\alpha N_t^{1-\alpha}$$

$$N_{t+1} = N_t + Z (K_t)^\alpha N_t^{1-\alpha}$$

$$1 + g_{t+1}^N = 1 + Z k_t^\alpha$$

so we have

$$g_{t+1}^N = t_t$$ (21)

and

$$t_t = Z k_t^\alpha$$ (22)

$$I_t = \int_0^1 I_t^j \, dj = \pi r_t K_t + \pi \rho_{t+1} [a_{t+1} (\xi (1 - \delta)) K_t + B_{t+1}]$$
Since $K_{t+1} = (1 - \delta) K_t + I_t$,

$$(1 + g^N_{t+1}) k_{t+1} = (1 - \delta) k_t + \pi r_t k_t + \pi \rho_{t+1} [a_{t+1} (\xi (1 - \delta)) k_t + (1 + g^N_{t+1}) b_{t+1}]$$

(23)

From the result of firm $j$’s R&D work, we can get

$$r_t = \alpha (Z \eta_t)^\frac{1}{\sigma} \left( \frac{w_t}{1 - \alpha} \right)$$

(24)

Goods market clearing condition implies

$$c_t + X_t + \pi r_t k_t + \pi \rho_{t+1} [a_{t+1} (\xi (1 - \delta)) k_t + (1 + g^N_{t+1}) b_{t+1}] = AX_t^\sigma$$

(25)

Thus, equations (1), (2), (10), (11), (13), (15), (20), (21), (22), (23), (24) and (25) alone with transversality conditions consist of a dynamic system which characterize the detrended equilibria of our model.

4 Balanced Growth Path

In this section, we derive and compare balanced growth path (BGP) of two cases. The first one is the case when there is no bubble while the second one is the case with bubbles. We have detrended all variables in last sector so variables should be at steady state alone BGP. We use detrended variables without time subscript to denote the steady state of these variables. Alone BGP, we know that $g^N = g^C$. We will use $g^N$ as a substitute when there is $g^C$ for convenience.

We first show that in both cases, given $q$ and $r$, capital $k$ and growth rate $g^N$ are determined in same way.

**Proposition 3** $k$ is determined implicitly by equation

$$rk = Z \left( \frac{1 - \sigma}{\sigma} \right) \sigma \tau^2 A^{1-\sigma} \frac{1}{1 - \beta (1 + Zk^\alpha)^{-\alpha}} k^\alpha - (1 - \alpha) r^\alpha \alpha \tau^\alpha \alpha \left[ Z \left( \frac{1 - \sigma}{\sigma} \right) \sigma \tau^2 A^{1-\sigma} \frac{1}{1 - \beta (1 + Zk^\alpha)^{-\alpha}} \right]^{\frac{1}{1-\alpha}}$$

(26)
and 

\[ g^N = Z(k)^a \]  

(27)

**Proof.** (27) is the direct result of (21) and (22). We only need to get (26) then growth rate is determined by \( q, a \) and \( r \). From (10),

\[ \rho = \beta \frac{1}{(1 + g^N)} = \beta (1 + Zk^a)^{-1} \]

This result alone with (2) give

\[ \eta = \left( \frac{1-\sigma}{\sigma} \right) \sigma \frac{r^2}{\sigma} A^{\frac{1}{\sigma}} \frac{1}{1 - \rho} = \left( \frac{1-\sigma}{\sigma} \right) \sigma \frac{r^2}{\sigma} A^{\frac{1}{\sigma}} \frac{1}{1 - \beta (1 + Zk^a)^{-1}} \]

From (24) we have

\[ w = (1 - \alpha) r^{\frac{\alpha}{\alpha - 1}} A^{\frac{\alpha}{\alpha - 1}} \left[ Z \left( \frac{1-\sigma}{\sigma} \right) \sigma \frac{r^2}{\sigma} A^{\frac{1}{\sigma}} \right]^{\frac{1}{\alpha - 1}} \]

Substitute these results into (11) we get (26). ■

### 4.1 Bubbleless BGP

In bubbleless equilibrium we know \( b_t = 0 \). From (20) we know that if there is no bubble in one period, there is no bubble for all periods. Thus equation (20) becomes an identity. At the same time, (23) becomes

\[ (1 + g^N_t k_{t+1} = (1 - \delta) k_t + \pi r_t k_t + \pi \rho_{t+1} [a_t + a (1 - \delta)] k_t \]  

(28)

**Proposition 4** When credit constraints are binding, capital price \( q_t \) capital return \( r_t \), detrended capital \( k_t \) and growth rate \( g_t^N \) is determined by

\[ 1 + g^N = (1 - \delta) + \pi r + \pi q (\xi (1 - \delta)) \]  

(29)

\[ \frac{q(1 + g^N)}{\beta} = r + q (1 - \delta) + \pi (q - 1) (r + q \xi (1 - \delta)) \]  

(30)

alone with (26) and (27).
Proof. We get (29) by substituting (15) into (28). By (15) and (13) we have (30). (29), (30) alone with (26) and (27) we derive from proposition 2, we get a four variables equations system which give us $q_l$, $r_l$, $k_l$ and $g_l$. ■

Unfortunately, it is impossible to derive the analytical solution of the variables. Thus, later we cannot compare the results between bubbleless BGP and bubbly BGP directly. However, we can use numerical method to check the results.

4.2 Bubbly BGP

We now study bubbly BGP. Here $b_t \neq 0$ and we cannot omit the (20). Just like what we have discussed in Bubbleless BGP, next proposition gives us the result of capital price $q$ capital return $r$, detrended capital $k$ and growth rate $g^N$. We use $q^b$, $r^b$, $k^b$ and $g^N_b$ as denotation.

Proposition 5 When there are credit constraints,

$$q^b = \frac{1 - \beta + \pi \beta}{\pi \beta}$$

$r^b$, $k^b$ and $g^N_b$ are determined by

$$\frac{q(1 + g^N)(1 - \beta + \pi \beta)}{\pi \beta^2} = r + \frac{(1 - \beta + \pi \beta)(1 - \delta)}{\pi \beta}$$

$$+ \frac{1 - \beta}{\beta} \left( r + \frac{1 - \beta + \pi \beta}{\pi \beta} \xi (1 - \delta) \right)$$

alone with (26) and (27).

Proof. Consider the case alone BGP. (20) gives us $q^b$. By (15) and (13) we have

$$\frac{q(1 + g^N)}{\beta} = r + q (1 - \delta) + \pi (q - 1) (r + q \xi (1 - \delta))$$

. Plug $q^b$ into it we have (31). (31), (26) and (27) are a three variables three equations system. ■

It is easy to check that $q^b > 1$ which means when collateral constraints are binding, bubbles always exist and the existence of bubbles never totally
eliminate the effects of collateral constraints.

4.3 Compare Bubbly BGP with Bubbleless BGP

According to proposition 2, 3 and 4, we know how to get values of variables on BGP. Unfortunately, it is impossible to get most results analytically for the system is too complicated. We use numerical method alone with equations we derive before to discuss how bubbles affect R&D and how credit constraints affect bubbles. The parameters are reported in table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\beta$</th>
<th>$\delta$</th>
<th>$\pi$</th>
<th>$\xi$</th>
<th>$\alpha$</th>
<th>$\sigma$</th>
<th>$A$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.95</td>
<td>0.15</td>
<td>0.04</td>
<td>0.5</td>
<td>0.3</td>
<td>0.99</td>
<td>1</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table 1: Values of Parameters

We can get both bubbly BGP and bubbleless BGP. Some important variables alone the BGP are reported below in table 2. Here $e$ is detrended debt. Compared with BGP without bubble, bubbly BGP has higher detrended capital level. Thus, the growth rate of bubbly BGP is greater. At the same time, capital price $q$, relationship between capital and equity value $a$ and capital return rate $r$ decrease. The detrended stock value of bubbly BGP, however, is less than the bubbleless one. To understand these phenomena, we first review the detrended capital transition (23)

$$(1 + g_{t+1}^N)k_{t+1} = (1 - \delta)k_t + \pi r_t k_t + \pi r_{t+1} \left[ a_{t+1} \left( \xi (1 - \delta) \right) k_t + (1 + g_{t+1}^N)b_{t+1} \right]$$

From transition function, we know that capital increases if revenue increases which means $\pi r_t k_t$ increases or firms have more access to external funding which means $\rho_{t+1} \left[ a_{t+1} \left( \xi (1 - \delta) \right) k_t + (1 + g_{t+1}^N)b_{t+1} \right]_{t+1}$ increases.

Bubbles increase discounted stock price if lenders take over the firms thus provide more collateral to help reduce liquidity mismatch. This direct effect
increases investment in R&D sector so it is the positive effect on growth rate. This positive effect can be called crowd in effect like other literatures (Hirano and Yanagawa 2017, Miao and Wang 2018) about bubbles. If we only consider this direct effect, bubbles certainly help R&D and growth.

However, there is also some indirect effects in general equilibrium which offset some of the positive effects. First of all, capital in our model not only be used to create new patents, but also be used to increase cum-dividend equity value so that when they have investment opportunities they can borrow more. Since bubbles in stock price have the same effect, demand of capital decreases which decreases price of capital $q$. That’s why we see in both examples $q$ and $a$ drop significantly when there are bubbles. This negative effect offsets some positive effects especially when credit constraints are not binding very much.

It is worth mention that this direct effect does not ensure higher detrended stock price of bubbly BGP. This is because indirect effects reduce the stock price by reducing $a$ at the same time bubbles increase the stock price. Sometimes the indirect effects are not too big so the stock price still increases while sometimes the indirect effects are big enough so the stock price may decrease. However, collateral constraints are related with stock price if lenders take over and this stock price always increases. This is the just we show in figure 1 in introduction.

That is not the end of the story. The effects we discuss above only ensure alone bubbly BGP firms get more loans. Since capital price $q$ decreases, return of capital $r$ also decreases which means that even with same amount of capital firm get less revenue through innovation activities and has to decrease the investment. Though there are these two negative effects which offset some of the positive effect, the positive effect is always the dominant one so stock price bubbles always encourage investment in innovation sector and increase growth rate. We can see this result when we do robust check. It is worth to mention that these direct and indirect effects are very similar with Oliver’s finding.
4.4 Robustness of our model

In some literature about bubbles, bubbles may only exist in some economies satisfy some certain conditions. Hirano and Yanagawa (2017) find that bubbles only exist when an economy had intermediate financial frictions in their model. Thus, this kinds of bubble region restrict the usefulness of the model.

Although it is impossible to derive the conditions under which bubbles exist analytically, we can use numerical way to show that bubbly BGP is rather robust so under most situations bubbles may exist. We also show that the result bubbly BGPs always have higher growth rate is robust. What is more, the tighter credit constraints are, the more benefit bubbles bring.

Our study focuses on two parameters $\xi$ and $\pi$. $\xi$ reflects the taking over cost when default. The smaller the $\xi$ is, the more costly the taking over is. Thus, credit constraints are tighter. During the robust test, we assume other parameters will have values the same as the study in previous subsection.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>1</th>
<th>0.7</th>
<th>0.3</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b/V(k)$</td>
<td>49.54%</td>
<td>54.69%</td>
<td>60.15%</td>
<td>62.39%</td>
</tr>
<tr>
<td>$g$</td>
<td>4.1%</td>
<td>4.0%</td>
<td>3.9%</td>
<td>3.8%</td>
</tr>
</tbody>
</table>

Table 3: Robustness check when credit constraints are tighter

Table 3 is the result showing how different $\xi$ affect bubbles. $b/V(k)$ characterize the average size of bubbles compare with firms’ value. When $\xi$ decreases, credit constraints are tighter and tighter. Growth rate decreases and bubbles increase. Even when $\xi = 0.1$ which means lenders can only take over 10% of original capital when borrowers default, bubbles still exist. This test means our model is very robust on financial conditions. Bubbles is possible to exist even in an economy with extremely tight credit constraints.

Besides $\xi$, $\pi$ is another parameter we have interested in. $\pi$ is the probability a firm find an investment opportunity in one period. If the probability is higher, more firms have investment opportunities thus the economy is more efficient in reallocating social resource. Thus the credit constraints are not binding so tight as before. For this reason, bubbles are shrinking quickly with the increasing of $\pi$. This can be seen in numerical analysis.
Table 4: Robustness check when probability of investment opportunities increase

<table>
<thead>
<tr>
<th></th>
<th>0.003</th>
<th>0.04</th>
<th>0.1</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b/V(k)$</td>
<td>70.37%</td>
<td>57.59%</td>
<td>37.94%</td>
<td>11.89%</td>
</tr>
<tr>
<td>$g$</td>
<td>1.9%</td>
<td>4.6%</td>
<td>4.7%</td>
<td>4.9%</td>
</tr>
</tbody>
</table>

What is the relationship when both $\xi$ and $\pi$ change? We give a region in which bubbly BGP are possible to exist. As usual, other parameters are the same as before. As shown in figure 3, the shade area is the region where bubble are possible to exist. With the decreasing of $\xi$, bubbles may exist in economy with higher chance of investment opportunities. This is because decreasing of $\xi$ makes credit constraints tighter. Even there are a large number of firms can invest, they still want to borrow more. From figure 3, we find that bubbles are possible to exist in many different cases which

![Figure 3: Region in which bubbly BGPs are possible to exist.](image-url)
means our model is more useful than Hirano and Yanagawa (2017). Figure 4 gives us how $\pi$ and $\xi$ affect bubbles. Generally speaking, if $\pi$ decreases of $\xi$ decreases, credit constraints are tighter, thus value of bubbles are greater.

Figure 5 gives us the information on the difference of growth rate between bubbly BGP's and bubbleless BGP's. Bubbly BGP's always have higher growth rate. At the same time, the smaller the $\pi$ and $\xi$ are, the bigger the difference it is. The reason is very simple, smaller $\pi$ and $\xi$ increase the tightness of credit constraints so bubbles play more important role in the economy.

5 Dynamics

In this section, we study the dynamics of the model. We first study the transition around bubbly BGP. We then study what happens when bubbles burst unanticipatedly. The stochastic burst of bubbles will be studied in next
section where we extend our baseline model into a stochastic model.

5.1 The dynamics around bubbly BGP

Since we have a big dynamic system, we are unable to derive analytical results for local dynamics. However, we can solve it numerically. We find rank conditions are satisfied for our examples. As in figure 6, we start from the point where detrended capital is about 10% more than alone BGP path. At this point, detrended bubbles is about 2.8% smaller than the BGP bubbles. With time going on, detrended capital is decreasing while detrended bubbles are increasing. In the end, they converge to the level of those alone BGP with bubbles.
Figure 6: Dynamics when there are 10% more capital than alone bubbly BGP

5.2 Unanticipated burst of bubbles

One of the most obvious feature of bubbles are bubbles tend to burst. For example, when Dot-com bubbles burst suddenly, Nasdaq Composite index fell 25% in one week. The price of Bitcoin dropped from around $19000 to around $6000 in less than two months. (He 2018) Very few people realize the bubbles is going to burst before it really happens. There are two ways to deal with burst of bubbles. The first approach is bubbles will burst unanticipatedly. However, many economists believe that although people do not know when bubbles burst, they expect bubbles will burst sooner or later. Thus the second approach is stochastic bubble. We study how unanticipated burst of bubbles have effects on the economy in our model. In next section, we study what happens when bubbles burst stochastically.

We assume the economy is growing alone bubbly BGP when there is an unanticipated shock at period 2. The shock changes the consensus that people believe others will not accept the overvaluation of equity. Thus, bubbles burst and there is no bubble from that periods on. Figure 7 is the result of what happens when bubbles burst.

From figure 7 we can see what happens with other variables when bubbles
burst. Since there is no bubble any more, credit constraints bind tighter and firms cannot get so much loans as before. Firms have to reduce their investment which leads to the decreasing of detrended capital from period 3. Capital gradually converges to the level of bubbleless BGP. Growth is also slower because innovation is slowed with the limit of capital. Capital price, however, jump at the time of bubbles burst and then grows slowly. The jump of price is due to the jump of capital demand since capital is now the only instruments to be used to increase the value of collateral. After that, amount of capital is decreasing which leads to the scarcity of capital which drives the price up gradually and also increases capital return rate. Return of capital $r$ increases. When bubbles burst, the jump of capital price increases capital return rate immediately. After that, capital return rate goes up gradually with the increasing of capital and capital price. $\alpha$ increases following the same pattern of capital price. Thus, economy will converge to bubbleless
BGP gradually if bubbles burst unanticipatedly.

6 Extensions

In this section, we study two extensions of our model. In the first extension, we introduce stochastic bubbles into our model. Up to now, bubbles are deterministic and they only burst when there is an unanticipated shock. People do not believe bubbles may burst unless it really happens even though they admit there are bubbles in stock market. These assumptions are not very realistic. Though it is hard to know when bubbles burst, people believe bubbles will burst in the future and the probability of burst affects people’s decision. To reflect this, we assume bubbles may burst every periods with a probability.

The second extension is studying how bubbles affect resource allocation. In the baseline model, we focus on how bubbles affect R&D sector directly. However, bubbles in one sector may have reallocation effects of resource. In the second extension, we assume both final product sector and R&D sector must use labor to produce and labor flow from one sector to the other sector freely. Thus, bubbles in R&D sectors can reallocate labor.

6.1 Stochastic bubbles

We assume if there are bubbles, they may burst at probability $\theta$ every period. The burst will happen before R&D activities. Other setting are the same as baseline model. If bubbles burst, everything works like the bubbleless equilibrium in the baseline model. We only need to consider the case when there are bubbles. Everything is the same as baseline model except (5) becomes

$$V_t(K_i) = (1 - \pi)(1 - \theta) \max_{K_{i+1, B_i}} [D_i^j + \rho_{t+1}V_{t+1}(K_{i+1}^j)]$$

$$+ \pi (1 - \theta) \max_{K_{i+1, B_i}^j} [D_{It}^j + \rho_{t+1}V_{t+1}(K_{It+1}^j)] + \theta V_{t}^{#}(K_i)$$

(32)
where \( V_t^# \left( K_t^j \right) \) is the bubbleless cum-dividend equity value we have derived before. We show the next proposition in Appendix

**Proposition 6** When there are bubbles, \( V_t \left( K_t^j \right) = a_t K_t^j + B_t \) where

\[
a_t = (1 - \theta) r_t + (1 - \theta) q_t (1 - \delta) + (1 - \theta) \pi (q_t - 1) \left[ r_t + \rho_{t+1} a_{t+1} \xi (1 - \delta) \right] + \theta a_t^#
\]

(33)

\[
B_t = (1 - \theta) [1 + \pi (q_t - 1)] \rho_{t+1} B_{t+1}
\]

(34)

We can see the stochastic bubbles model has the similar result with baseline model. The only difference between (33), (34) and (13), (14) are the probability of burst. There is \( 1 - \theta \) chance that bubbles still exist at the beginning of that period thus we have terms similar like before. However, there is probability \( \theta \) that bubbles burst. If bubbles do burst, firms will operate as the firms in bubbleless equilibrium so we have the term \( \theta a_t^# \).

We then use numerical method to study the stochastic model.

Here we have \( \theta = 0.05 \) and all other parameters are same as the baseline model. The results are in figure 8. After the burst of bubbles, the path to the bubbleless equilibrium is similar with the unanticipated shock. The mechanism is the same as baseline model.

### 6.2 Reallocation effects model

We assume the household has labor supply \( \bar{L} \). Budget constraint of the representative household is

\[
C_t + \int (V_t^j - D_t^j) \psi_t^j dj = \int V_t^j \psi_t^j dj + W_t \bar{L}
\]

Final goods producer now has the technology

\[
Y_t = A \int_{n=1}^{N_t} (X_t^n)^\sigma \left( L_t^Y \right)^{1-\sigma} dn, 0 < \sigma < 1
\]

Here \( L_t^Y \) is the labor hired by final goods producer. Thus, only \( \bar{L} - L_t^Y \) labor works in R&D sector. Since most derivation and results are similar with
baseline model. We put all the derivation into appendix. We only provides the numerical results here.

Given the parameters in baseline model and $L = 2$, we have bubbly BGP and bubbleless BGP in table 5. The results are similar like we have discussed

<table>
<thead>
<tr>
<th>$b/V(k)$</th>
<th>$g^N$</th>
<th>$q$</th>
<th>$a$</th>
<th>$r$</th>
<th>$k$</th>
<th>$L^s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>bubbly</td>
<td>53.97</td>
<td>1.7%</td>
<td>2.32</td>
<td>2.48</td>
<td>0.44</td>
<td>0.0010</td>
</tr>
<tr>
<td>bubbleless</td>
<td>0</td>
<td>1.4%</td>
<td>8.22</td>
<td>8.77</td>
<td>0.60</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

Table 5: Values of variables alone bubbly and bubbleless BGP in reallocation effects model

in baseline model. The only difference is that labor works in final products sector in bubbly economy is less than the labor in bubbleless economy. Which means labor flow from final products sector to R&D sector. The intuition is simple. We have discussed that there are more capital in R&D sector when
there are bubbles thus the marginal productivity of labor in R&D sector is
higher. R&D firms would like to hire more labor to produce new blueprints.
This effect alone with the effects we discussed before increases the positive
effects of bubbles in R&D sector and helps the economic growth.

We also study the burst of bubbles. As the baseline model, the pattern
of burst both in unanticipated shock and stochastic bubbles show the similar
result. We will only report the unanticipated shock here in figure 9. The
stochastic bubbles are reported in appendix.

Figure 9: Unanticipated burst of bubbles in reallocation effects model

Most of the path are similar with baseline model. The only significant dif-
ference is when bubbles burst, growth rate first increases and then decreases
gradually in this model while growth rate decreases with the decreasing capital in baseline model. This may be a surprising result. Why does the economy grow faster when the bubbles burst and capital start decreasing? This is because the change of stochastic discount rate $\rho_t$. Since $\rho_t$ increases generally after the burst of bubbles, the price of patent $\eta_t$ also increases. Although capital decreases soon after the burst of bubbles, increasing of patent price increases the marginal productivity of labor in R&D sector thus it attracts more labor flows out of final goods sector and works for R&D sectors. This effect compensate the decreasing of capital. At the beginning, this reallocation effect is strong enough so there are more patents are produced. With less and less capital, marginal productivity of labor is decreasing and people flow out of R&D sector and the growth rate is smaller and smaller until it reaches the bubbleless BGP.

7 Conclusion

In this paper, we introduce credit constraints into a standard endogenous growth model with innovation. We show that there are multiple equilibria and stock price bubbles exist in one of the equilibria. To the extent of our knowledge, this is the only study about endogenous growth with bubbles by using a well defined R&D growth model and it is the only model in which bubbles are generated by features of innovation sectors. Our study finds that stock price bubbles in innovation sector encourage innovation and increase growth rate by reducing liquidity mismatch. Economy with tighter credit constraints benefit more from bubbles. Thus, it may be wise for governments not to make bubbles in innovation sector burst but use policy instruments to reduce financial frictions in innovation sector. This paper can be a bridge between traditional growth model and bubbles.

Besides the economic phenomenon we discussed in this paper. There are some extension which can be done in the future. Our paper has R&D sector and bubbles. At the same time, household buy shares of R&D firms. These are close to the situation the venture capitalists face. In the future, we can extend the R&D sector and this model may be used to study venture
capitalists’ behavior.

References


A Proof of proposition 6

Assume solution of R&D firm $j$’s problem is $V_t (K_t^j) = a_t K_t^j + B_t$. Substitute (6), (7), (8), (9) and the solution we guess into (32) we have

$$a_t K_t^j + B_t = \max_{K_{t+1}^j, K_{t+1}^{j+1}, I_{t+1}^j, B_{t+1}^j} \left[ (1 - \theta) r_t K_t^j + (1 - \theta) q_t (1 - \delta) K_t^j \\
(1 - \theta) \rho_{t+1} B_{t+1} + (1 - \theta) (1 - \pi) \left[ -q_t K_{t+1}^j + \rho_{t+1} a_{t+1} K_{t+1}^j \right] \\
+ (1 - \theta) \pi \left[ (q_{t+1} - 1) \left( r_t K_t^j + \rho_{t+1} a_t K_{t+1}^j \right) - q_t K_{t+1}^j + \rho_{t+1} a_{t+1} K_{t+1}^j \right] \\
+ \theta a_t^# K_t^j \right]$$
By taking first order derivative of $K_{t+1}^j$ we have

$$q_t = \rho_{t+1}a_{t+1}$$

By comparing the left hand side and right hand side we get

$$a_t = (1 - \theta) r_t + (1 - \theta) q_t (1 - \delta) + (1 - \theta) \pi (q_t - 1) [r_t + \rho_{t+1}a_{t+1} \xi (1 - \delta)] + \theta a_t^#$$

$$B_t = (1 - \theta) [1 + \pi (q_t - 1)] \rho_{t+1} B_{t+1}$$

which are the proposition.

**B  Derivation of reallocation effects model**

**B.1 Households**

The only difference is the budget constraints are now

$$C_t + \int (V_t^j - D_t^j) \psi_t^j d_j = \int V_t^j \psi_t^j d_j + W_t L$$

**B.2 Final goods producer**

Since now the technology is

$$Y_t = A \int_{n=1}^{N_t} (X_t^n) \sigma (L_t^Y)^{1-\sigma} dn, \ 0 < \sigma < 1$$

profit maximization problem of the final goods producer is

$$\max A \int_{n=1}^{N_t} (X_t^n)^\sigma (L_t^Y)^{1-\sigma} dn - \int_{n=1}^{N_t} P^n_t X_t^n dn - W_t L_t^Y$$

subject to the production function. It is easy to solve profit maximization problem and we have the demand function for intermediate goods $n$

$$P^n_t = \sigma A (X_t^n)^{\sigma - 1} (L_t^Y)^{1-\sigma}$$
\[ W_t = (1 - \sigma) A \int_{n=1}^{N_t} (X^n_t)^{\sigma} (L^n_t)^{-\sigma} \, dn \]

### B.3 Intermediate goods producers

Producer of Intermediate good \( n \) has profit

\[ (P^n_t - 1) X^n_t = \sigma A (X^n_t)^{\sigma} (L^n_t)^{1 - \sigma} - X^n_t \]

Since we have already had intermediate goods \( n \)'s demand function, we can find the price intermediate goods producer of goods \( n \) set.

\[ P^n_t = \frac{1}{\sigma} \]

and the amount the producer produces

\[ X^n_t = \sigma^{\frac{1}{1-\sigma}} A^{\frac{1}{1-\sigma}} L^n_t \quad (35) \]

Then the profit of producing goods \( n \) every period is

\[ \left( \frac{1 - \sigma}{\sigma} \right) \sigma^{\frac{2}{1-\sigma}} A^{\frac{1}{1-\sigma}} L^n_t \]

Since we know that it is competitive monopolistic market, the discounted total profits from selling goods \( n \) must be equal to the cost of buying patent to produce goods \( n \), which means

\[ \sum_{s=t}^{\infty} \rho(s, t) \left( \frac{1 - \sigma}{\sigma} \right) \sigma^{\frac{2}{1-\sigma}} A^{\frac{1}{1-\sigma}} L^n_t = \eta_n \]

Here \( \rho(s, t) = \prod_{v=t+1}^{s} (\rho_{v+1}) \) if \( s \neq t \), \( \rho(s, t) = 1 \) if \( s = t \). Since only variables in \( \eta_n \) are time variables, patents created in the same period have the same price. This result gives us

\[ \sum_{s=t}^{\infty} \rho(s, t) \left( \frac{1 - \sigma}{\sigma} \right) \sigma^{\frac{2}{1-\sigma}} A^{\frac{1}{1-\sigma}} L^n_t = \eta_n = \eta_t \quad (36) \]
B.4 R&D Sector

Same as baseline model.

B.5 Competitive Equilibrium

**Definition 7** A competitive equilibrium is defined as allocations

\[ \{Y_t, K_t, C_t, I_t, N_t, E^j_t, T_t, L^Y_t, I^j_t, K^j_t, T^j_t, Y^j_t, \psi^j_t, X^n_t\} \] and prices

\[ \{w_t, P^i_t, R^j_t, q_t, \eta_t, r_t, V^j_t\} \] such that household maximize its utility and firms in all three sectors maximize their profits and market clearing conditions are satisfied which are stock market is clearing \( \psi^j_t = 1 \), labor market is clearing \( \int_0^1 L^j_t dj + L^Y_t = \bar{L} \), debt market is clearing \( \int_0^1 E^j_t dj = 0 \), capital market is clearing \( K_{t+1} = (1 - \delta) K_t + I_t \), goods market are clearing \( C_t + \int_{n=0}^{N_t} \int_0^1 X^n_t d\eta + I_t = Y_t \) and the amount of patent follows \( N_{t+1} = N_t + T_t \).

B.6 Detrended dynamic system

The detrended dynamic system now becomes

\[
X_t = \sigma^{\frac{2}{\sigma}} A^\frac{1}{\alpha} L^Y_t
\]

\[
\sum_{s=t}^{\infty} \rho(s, t) \left( \frac{1 - \sigma}{\sigma} \right)^{\frac{2}{\sigma}} A^\frac{1}{\alpha} L^Y_t = \eta_n = \eta_t
\]

\[
\rho_{t+1} = \beta \frac{c_t}{c_{t+1} (1 + g_{t+1})}
\]

\[
r_t k_t = Z \eta_t (k_t)^{\alpha} - w_t (\bar{L} - L^Y_t)
\]

\[
a_t = r_t + q_t (1 - \delta) + \pi (q_t - 1) (r_t + \rho_{t+1} a_{t+1} (1 - \delta))
\]

\[
q_t = \rho_{t+1} a_{t+1}
\]

\[
b_t = [1 + \pi (q_t - 1)] \rho_{t+1} (1 + g_{t+1}) b_{t+1}
\]

\[
g_{t+1}^N = t_t
\]

\[
t_t = Z k_t^\alpha (\bar{L} - L^Y_t)^{1-\alpha}
\]
We also study the case when bubbles burst stochastically. The setting is similar with stochastic bubbles in baseline model. We only report the simulation result here in figure 10. Just as the relationship between stochastic burst and unanticipated burst in baseline model, the pattern of stochastic burst and unanticipated burst are similar. The intuition is also similar with intuition we discuss before.

\begin{align*}
(1 + g_t^N)k_{t+1} &= (1 - \delta) k_t + \pi r_t k_t + \pi \rho_{t+1} \left[ a_{t+1} (\xi (1 - \delta)) k_t + (1 + g_t^N b_{t+1}) \right] \\
r_t &= \alpha \left( Z \eta_t \right)^{\frac{1}{\alpha}} \left( \frac{w_t \left( \bar{L} - L_t^Y \right)^{\alpha}}{1 - \alpha} \right)^{\frac{\alpha - 1}{\alpha}} \\
c_t + X_t + \pi r_t k_t + \pi \rho_{t+1} \left[ a_{t+1} (\xi (1 - \delta)) k_t + (1 + g_t^N b_{t+1}) \right] &= AX_t^\sigma \\
w_t &= (1 - \sigma) A (X_t)^{\sigma} (L_t^Y)^{-\sigma}
\end{align*}

\section*{B.7 Stochastic bubbles burst}

We also study the case when bubbles burst stochastically. The setting is similar with stochastic bubbles in baseline model. We only report the simulation result here in figure 10. Just as the relationship between stochastic burst and unanticipated burst in baseline model, the pattern of stochastic burst and unanticipated burst are similar. The intuition is also similar with intuition we discuss before.
Figure 10: Burst of stochastic bubbles in reallocation effects model