Dynamic Reserves in Matching Markets∗

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Abstract

We study a school choice problem under affirmative action policies where authorities reserve certain fractions of the seats at each school to specific student groups, and students have preferences not only over the schools they are matched to but also what type of seats they receive. Such reservation policies might cause waste when there is low demand from some student groups. To propose a solution to this issue we construct a family of choice functions, dynamic reserves choice functions, for schools that respect within-group fairness and allow transferring otherwise vacant seats from low-demand groups to high demand-groups. We analyze the properties of the choice functions in this family. We propose the cumulative offer mechanism as an allocation function where each school uses a dynamic reserves choice function and show that it is stable with respect to schools’ choice functions, is strategy-proof, and respects improvements in school priorities. Furthermore, we provide comparative statics results on capacity transfer schemes one of which is that transferring more of otherwise vacant slots leads to strategy proof Pareto improvement under the cumulative offer mechanism.

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1 Introduction

The theory of two-sided matching and its applications have been studied since the seminal work of Gale and Shapley (1962). Nevertheless, many real-life matching markets are subject to constraints such as affirmative action in school choice, and both economists and policy-makers are often faced with new challenges from such constraints. Admission policies in school choice systems often use reserves to grant applications from certain backgrounds higher priority for some available seats. Engineering school admissions in India, for instance, function through a centralized matching market in which different privileged backgrounds (namely, scheduled castes (SC), scheduled tribes (ST), other backward classes (OBC) and general category (GC) applicants) are treated with different admissions criteria. Students have different preferences over which admission criteria they are admitted under and it is optional to reveal their background or not. Therefore, students may prefer not to reveal their caste and tribe information in the application process. Besides this strategic calculation burden on students, the current system suffers from a crucial market failure which is the main focus of this paper: The centralized assignment mechanism fails to transfer some unfilled seats reserved for under-privileged castes and tribes to the use of remaining students. Hence, it is vastly wasteful.

In this paper, we model this problem and propose a remedy through a matching model with contracts with the ability to utilize vacant seats of certain types for other students. Moreover, our remedy removes the strategic manipulation burden, about whether to declare their background or not, from students’ shoulders. We propose a strategy-proof and stable mechanism that respect improvements in school priorities in this framework.

More specifically, our model addresses the real-life applications as follows: There are schools and students to be matched. Each school initially reserves a certain number of its slots for different privilege groups (student types). A given student may possibly match with a given school under more than one type. Each school has a pre-specified sequence in which different sets of slots, each of which accept students in a single privilege group.

1If students belonging to SC, ST and OBC communities do not declare their background information they will be considered as GC applicants. Students who do not belong to SC, ST or OBC categories also considered only for GC seats. GC is actually a “no-privilege” category that every student belongs to. However, for our modeling purposes we consider GC as one of the privilege types by abusing the meaning of the word “privilege”. Hence, this is why we can say that each student is a member of at least one privilege group.

2The current practice of engineering school admissions in India and its matching-theoretical shortcomings are explained in detail in Aygün and Turhan (2017).

3Our remedy makes it a weakly dominant strategy for each student to report their caste and tribe information.

4We will call this sequence as precedence sequence which is different than precedence order of Komin-
ilege type, are to be considered. Different schools might have different orders. Since a student might have more than one privilege type, the set of students cannot be partitioned into privilege groups. Each student has a preference over school-privilege type pairs. Students care not only about which school they are matched to but also about the privilege type under which they are admitted. Each school has a target distribution of its slots over privilege types, but do not consider these target distributions as hard bounds. If there is less demand from at least one privilege type, schools are given the opportunity to utilize these vacant slots by transferring them over to other privilege types. Schools might have preferences over how to redistribute these unassigned slots. Each school has a complete plan where they state how they want to redistribute these slots, so we take capacity transfer schemes exogenously given. The only mild condition imposed on the capacity transfer scheme is monotonicity, which requires that (1) if more slots are left from one or more sets of slots, the capacity of the sets of slots considered later in the precedence sequence is required to be weakly higher, and (2) a school cannot decrease the total capacity in response to increased demand for some sets of slots.

We design choice functions for schools that allow them to transfer capacities from low-demand privilege types to high-demand privilege types. Each school respects an exogenously given precedence sequence between different sets of slots, where each set is associated with a single privilege type, when it fills its seats. Each school has a strict priority ordering (possibly different that other schools’) over all students from which priority orderings for different privilege types are derived in a straightforward fashion. For each set of slots there is an associated choice function, which we call a “sub-choice function”. For the case of IIT admission, we assume that these sub-choice functions are q-responsive. Given the target distribution of the school and a set of contracts, the first set of slots, which is associated with a specific privilege type (according to school’s precedence sequence), of the school fills its slots according to its sub-choice function. Then, it moves to the second sets of slot which is also associated with a specific privilege type. Sub-choice functions are linked to each other by two components. First, since we take a pre-specified precedence sequence, the choice in each set of slots depends on what has been chosen by the sets of slots that were considered earlier, according to the precedence sequence. Given the chosen contracts from the first set of slots, the

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5Hard bounds and soft bounds are analyzed in detail in Hafalır et al. (2013) and in Ehlers et al. (2014).

6Westkamp (2013) introduces this monotonicity condition on capacity transfer schemes.

7The definition of q-responsive functions will be given in Section 2.1. See Chambers and Yenmez (2017).
remaining sets of contracts for the second set of slots can be found as follows: if a student has one of her contracts chosen by the first set of slots, then all of her contracts are removed for the rest of the choice process. The second component that links sub-choice functions of different sets of slots, each of which is associated with a specific privilege type, is that the capacity of a set of slots changes dynamically according to the number of vacant slots in the previous sets of slots considered earlier according to the precedence sequence. The idea here is that the capacity of the set of slots following the first set of slots, according to the precedence sequence, is a function of the numbers of unassigned seats in the first set of slots. The capacity of the third set of slots is a function of the numbers of unassigned seats in the first and second sets of slots, and so on. In short, each sub-choice function has two inputs: the set of remaining contracts to consider, which depends on the choices of the sets of slots considered before it, and its capacity, which changes dynamically according to the number of unassigned seats in the sets of slots considered earlier. The overall choice of a school is the union of sub-choices by its different sets of slots.

After introducing our model and describing the family of dynamic reserves choice functions in Section 2, we define two stability notions in Section 3, group-stability and pairwise stability, in matching with contracts framework and show that these two stability notions are equivalent when each school employs a dynamic reserves choice function (Proposition 1). Then, in order to better explain what stability achieves in our setting, we characterize stability by following normative axioms: feasibility, no justified envy with other schools, no justified envy with own schools, and non-wastefulness up to the dynamic capacities (Proposition 2). In section 4, we analyze the properties of dynamic reserves choice functions. We show that every dynamic reserves choice function satisfies the irrelevance of rejected contracts (IRC) condition of Aygün and Sönmez (2003) (Proposition 3). Some dynamic reserves choice functions might fail unilateral substitutability (ULS) condition of Hatfield and Kojima (2010) and the law of aggregate demand condition of Alkan (2002) and Hatfield and Milgrom (2005) (Proposition 4 and Proposition 6). Each dynamic reserve choice function satisfies a weaker substitutability condition, bilateral substitutability (BLS), of Hatfield and Kojima (2010) (Proposition 5). We then utilize the theory of hidden substitutes of Hatfield and Kominers (2016) and show that each dynamic reserves choice function has a completion (Proposition 7) and this completion satisfies the IRC (Proposition 8), substitutability (Proposition 9) and the LAD (Proposition 10). Then, by invoking the results of Hatfield and Kominers (2016), we show that the cumulative offer mechanism is stable with respect to schools’

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8Alkan (2002) refers to the LAD as “cardinal monotonicity”.
9Biró, Fleiner, Irving and Manlove (2010) analyze a college admission model with common and upper quotas in their Section 5. The authors use choice functions for colleges that allow to select
dynamic reserves choice functions (Theorem 1) and \textit{(weakly) group strategy proof} for students (Theorem 3). We also show that the cumulative offer mechanism with respect to dynamic reserves choice functions \textit{respects unambiguous improvements}.

In Section 6, we define \textit{“more flexible”} criterion to compare two monotonic capacity transfer schemes given a precedence sequence. We say that a monotonic capacity transfer scheme $\tilde{q}$ is more flexible than monotonic capacity transfer scheme $q$ if $\tilde{q}$ transfers at least as many otherwise vacant seats as $q$ at every instance and there must be an instance where $\tilde{q}$ transfers strictly more otherwise vacant seats than $q$ does. We, then, define a \textit{choice function expansion} in a straightforward way. We say that a choice function $\tilde{C}$ is an expansion of $C$ if $\tilde{C}$ uses a more flexible capacity transfer scheme than $C$, all else equal. We show that when a choice function of a school is expanded in this way, from any given offer set, it chooses (weakly) higher number of contracts (Theorem 5). More importantly, when a choice function of a single school is expanded while choice functions of other schools remain unchanged, the outcome of the cumulative offer mechanism under the expanded profile of choice functions Pareto dominates the outcome of the cumulative offer mechanism under the original profile of choice functions, given the preference profile of students (Theorem 6). Theorem 6 is of particular importance because it describes a strategy-proof Pareto improvement.

1.1 Institutional Details Regarding Indian Institute of Technologies (IITs) Admissions

According to the \textit{“Reservation in India”} policy each engineering program in IITs reserves certain fractions of its slots to SC, ST and OBC students to level the playing field. These fractions vary across states. It is optional for students to declare their caste and tribe information. If a student from SC, ST or OBC communities does not declare his/her background information in the application process, then he/she is considered only for GC seats. Students who declare themselves as SC students are considered both for SC and GC seats by the programs. Students who declare themselves as ST students are considered both for ST and GC seats by the programs. Students who declare themselves as OBC students are considered both for OBC and GC seats by the programs. Students who do not belong to SC, ST or OBC categories and students who choose not to declare their background information are considered only for GC seats.

Students are ranked based on their total scores in the common entrance exam. Ties are broken by age so that each student has a unique ranking. Different programs
have different rankings because depending on the program the weights of mathematics, physics, chemistry and biology change to compute total test score. This determines the ranking for GC seats. Ranking for SC, ST and OBC slots are obtained from the GC ranking as follows: for each category, the relative ranking of the same-category students are preserved and the students from other categories are removed. Each student submits his/her preferences over programs. They can rank up to 100 programs.

The current allocation mechanism in the main round of admissions is a version of student-proposing deferred acceptance algorithm. Students apply to their top choice programs in the first stage. Each program first tentatively fills the GC slots following the GC ranking. If there are more students than the number of GC slots the remaining students are considered SC, ST and OBC seats depending on their background information revelation. For each category, students are tentatively assigned up to the capacity of that category following the ranking in that category. In general, each student who was rejected in the previous step applies to his/her next choice program. Each program tentatively fills its GC slots following the GC ranking among the tentatively held students from the previous step and new applicants. If there are more students that the number of GC slots, students are tentatively assigned to SC, ST and OBC slots depending on their caste revelation following the ranking in the relevant category. The remaining students are rejected. This algorithm ends in finitely many steps. When outcomes are announced, all students learn their program assignments together with the type of seat they are admitted under. The rationale to publicly announce what seat types students are admitted under is that the government wants to prove to the public that the reservation policy is indeed respected.

There are two major shortcomings of the procedure described above: (1) When there is low-demand from at least one of the privilege types the centralized assignment mechanism fails to transfer unfilled seats reserved for low-demand privilege type to high-demand privilege types. This causes a vast waste even though the number of students competing for seats is significantly larger than the number of available slots. (2) Even though students from SC, ST and OBC communities have preferences not only over programs but also over the type of seats they receive, the centralized assignment mechanism elicits their preferences only over programs. The narrow preference domain causes incentive issues and inefficiencies, as well as causes the allocation mechanism not to respect test score (priority) improvements of students. We refer the reader to Aygün and Turhan (2017) for details on these points, though we explain them shortly below.

1. The number of students from the SC, ST and OBC communities, who exceed the threshold score to be able to apply for IITS, are volatile over years. Since the reservations are set ex-ante and cannot be changed, in many programs, there
will be vacancies in SC, ST and/or OBC categories. In some states, such as Maharashtra, the government was able to change the law so that vacant seats within the OBC category, the largest reservation category, is transferred to the GC even though vacancies in SC and ST categories still go unfilled when there is low-demand from these groups. After the main round of admissions there is a secondary market in which the vacant OBC seats from the main round of admissions are allocated to unassigned GC applicants in the order of test scores. This is a well-intended attempt to reduce the waste. However, we argue that there is a better way to transfer vacant OBC seats into GC. If we modify the choice functions of programs in a way that vacancies in the OBC category is transferred into GC in the main rounds of admissions, the allocation mechanism will be non-manipulable and respect students’ test score improvements. We argue that the seats set aside for a type should be made available to other types, if they would otherwise go unused, within the main admission round.

2. There are two main reasons that students from designated communities care also about what type of seats they receive. First, in India, there is a stigma associated with receiving an affirmative action seats. Some students from designated communities prefer GC seats over affirmative action seats. Some do not even claim their caste/tribe privilege as the background information revelation is optional. Secondly, affirmative action seats come with financial advantages. In many states, receiving an affirmative action seat provides tuition waiver, free accommodation, etc. Hence, a GC seat and an affirmative action seats can be thought as different goods. Students who care more about financial benefits might prefer affirmative action seats over GC seats.

Reservation policy in India has been a hot political debate in India. There has been a negative stigma associated with a receiving an affirmative action seat. Our proposed design does not eliminate this stigma. However, it is crucial to note that part of the reason for hostility to the reservations of seats for historically discriminated groups stems from the egregious waste\textsuperscript{10} of seats that sometimes results under the present mechanism. In the famous 2006 Indian anti-reservation protests\textsuperscript{11} one of the demands made by the organizing group “Youth For Equality” was for “vacant positions in reserved government jobs to be thrown open for other eligible candidates.”\textsuperscript{12} This particular demand of the Youth For Equality is in line with what we suggest in our solution.

\textsuperscript{10}We thank an anonymous referee to make this important point.
\textsuperscript{11}“This movement was triggered by the staff and students of IITs.” Received from https://www.quora.com/Why-did-anti-reservation-campaign-fail-in-India-in-2006 (accessed on 10/15/2018).
\textsuperscript{12}Received from https://en.wikipedia.org/wiki/2006_Indian_anti-reservation_protests (accessed on 10/15/2018).
1.2 Other Applications: Extended Cadet-Branch Matching Problem

Section 5 of Sönmez and Switzer (2013) presents an extended model of cadet branch matching problem in which there are more than two terms (or “costs”) according to which a cadet might be matched to a branch. Each branch has a priority order over cadets and priority order of a term in a given branch is obtained from the branch’s priority order in a straightforward fashion. Branches have a target distribution of their slots over different terms. In this framework, the authors suggests a specific choice function for branches using nested multilayer priority structure. The nested multilayer priority structure choice function uses a specific precedence sequence of terms and specific capacity transfer scheme for branches. The extended model of cadet-branch matching problem and the choice function from nested multilayer priority structure can be formulated in our framework. Moreover, in our framework one can design different choice functions that use different precedence sequences and capacity transfer schemes so that different policy goals for the army can be implemented.

1.3 Related Work

The school choice problem was first introduced by seminal papers Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003). In Abdulkadiroğlu and Sönmez (2003), the authors also introduced a simple affirmative action policy with type-specific quotas. Kojima (2012) investigates the consequences of affirmative action policy with type-specific quotas on students’ welfare in a setup where there are only two types of students, minorities and majorities, and quotas for majority students only. He provides examples in which all minority students are made worse off under this type of affirmative action and he concludes that authorities should be cautious when implementing such affirmative action policies. To circumvent inefficiencies caused by majority quotas, Hafalır, Yenmez and Yıldırım (2013) offer minority reserves. Schools assign minority reserves such that if the number of minority students in a school is less than its minority reserves, then any minority student is preferred to any majority student in that school. If there are not enough minority students to fill the reserves, majority students can still be admitted to fill up that school’s reserved seats so that unfilled minority seats are allowed to be transferred to majorities. Hafalır, Yenmez and Yıldırım (2013)\textsuperscript{13} is the first paper that suggests dynamic reserves in a simpler setup.

Our paper studies a variant of the matching with complex constraints problem, introduced by Westkamp (2013), that seeks to control the distribution of assigned students.\textsuperscript{13}

\textsuperscript{13}See also Ehlers, Hafalır, Yenmez and Yıldırım (2014).
We allow each student to have more than just one type (privilege) whereas most of the controlled school choice literature, as in Westkamp (2013), assumes that each student has one type. We propose a controlled school choice model where students have strict preferences over type-specific matches with schools, especially to reflect the IITs admissions problem. Allowing for multiple types per student is needed for this to be a non-trivial innovation and capture the IITs admissions problem, and differentiates this paper from Westkamp (2013). This highlights a novel market design problem where the affirmative action category by which a student is admitted may actually be a concern to her, and is beyond the scope of Westkamp (2013). It is important to note that we adopt the capacity transfer scheme of Westkamp (2013) to this type of model and show that the resulting choice functions ensure that the cumulative offer algorithm defines a strategy-proof and stable mechanism. We provide a comparative statics result on transfer schemes, which does not have a counterpart in Westkamp (2013): more flexible transfers lead to a Pareto improving mechanism. The comparative statics result is especially valuable since it provides a normative foundation for recommending a more flexible interpretation of type-specific quotas.

Biró, Fleiner, Irving and Manlove (2010) study two important problems in the context of Hungarian college admission: college admissions problem with lower and upper quotas and college admission problem with upper and common quotas. The IITs admissions problem is very much related to the college admission problem with upper and common quotas that they discuss in Section 5 of their paper. The authors focus on existence of stable allocation and structural properties of stable matchings in their framework. However, our main focus is to provide a strategy-proof Pareto improvement by designing a family of choice functions that consider capacities of different groups of seats in each school as flexible and introducing a notion of “more flexible” capacity transfer scheme.

Our paper is also related to the research agenda on matching with constraints studied in a series of papers Kamada and Kojima (2015, 2017a, 2017b, 2018a, 2018b), Kojima, Tamura and Yokoo (2018), Goto et. al. (2017). In these papers, constraints are imposed on subsets of institutions as a joint restriction as opposed to constraints at each individual institution. A leading example in these papers is the medical match problem in Japan, in which the government imposes regional caps which is an upper bound on the number of doctors who can be placed in each region of the country. Our stability notion is very much related to the weak stability notion of Kamada and Kojima (2017a). Our comparative statics results on monotone capacity transfer schemes and respecting unambiguous improvements property distinguish our work from the research agenda on matching with constraints studied in the aforementioned papers.

The matching problem with dynamic reserves choice functions is a special case of
the matching with contracts model of Fleiner (2003) and Hatfield and Milgrom (2005).14 Fleiner (2003)15 introduced a matching model with contracts in which institutions are allowed to select multiple contracts with a given agent. Hatfield and Milgrom (2005) assumes substitutes condition which later on was weakened by Hatfield and Kojima (2010)16 while stability and strategy-proofness of the cumulative offer mechanism are maintained. The analysis and results of Hatfield and Kominers (2016) are the technical backbone of our results regarding stable and strategy-proof mechanism design as we show that every dynamic reserves choice function has a completion that satisfies the IRC condition of Aygün and Sönmez (2013), substitutability and the law of aggregate demand.

Kominers and Sönmez (2016) introduce another prominent family of choice functions in order to implement diversity objectives in many-to-one setting. In Section 4, we explain the relationship between the dynamic reserves choice functions family and the slot-specific priorities choice functions family of Kominers and Sönmez (2016).

Another very closely related paper is Hatfield, Kominers and Westkamp (2017a). The authors show how to model firms with cross-division constraints using the framework of matching with contracts. They introduce a model of hospital choice in which each hospital has a set of divisions and flexible allotment of capacities to those divisions that varies as a function of the set of contracts available. Building on the novel observable substitutability theory of Hatfield, Kominers and Westkamp (2017b), they show that the cumulative offer mechanism with respect to these flexible allotment choice functions is stable and strategy-proof.

Our paper is also related to Echenique and Yenmez (2015). The authors characterize choice functions for schools that regard students as substitutes while expressing preferences for a diverse student body. Dynamic reserves choice functions might seem similar to a family of choice functions the authors analyze: choice rules generated by reserves for priority a given priority profile. Dynamic reserves choice functions chooses contracts whereas choice rules generated by reserves chooses students. The difference is crucial because in our setting students have strict preferences not only over schools but rather over the cartesian product of schools and privilege types they can claim.

Nguyen and Vohra (2018) study matching problems with distributional constraints from a different perspective. In prior work, distributional constraints are formulated by imposing side constraints whose “right hand sides” are absolute numbers specified

14Echenique (2012) has shown that under the substitutes condition, which is throughly assumed in Hatfield and Milgroum (2005), the matching with contracts model can be embedded within the Kelso and Crawford (1982) labor market model. Kelso and Crawford (1982) built on the analysis of Crawford and Knoer (1981).
15Fleiner’s results cover results of Hatfield and Milgrom (2005) regarding stability. However, Fleiner (2003) does not analyze incentives.
16See also Hatfield and Kominers (2008).
2 Model

There is a finite set of students $I = \{i_1,...,i_n\}$, a finite set of schools $S = \{s_1,...,s_m\}$, and a finite set of student privileges (types) $T = \{t_1,...,t_p\}$. We call $T_i \subseteq T$ the set of privileges that student $i$ can claim and $\mathbf{T} = (T_i)_{i \in I}$ the profile of types that students can claim. We define $X_i = \{i\} \times S \times T_i$ the set of all contracts associated with student $i \in I$. Let $X = \bigcup_{i \in I} X_i$ be the set of all contracts. Each contract $x \in X$ is between a student $\mathbf{i}(x)$ and a school $\mathbf{s}(x)$ and specifies a privilege $\mathbf{t}(x) \in T_{\mathbf{i}(x)}$. There may be many contracts for each student-school pair. We extend the notations $\mathbf{i}(\cdot)$, $\mathbf{s}(\cdot)$ and $\mathbf{t}(\cdot)$ to the set of contracts, for any $Y \subseteq X$, by setting $\mathbf{i}(Y) \equiv \bigcup_{y \in Y} \{\mathbf{i}(y)\}$, $\mathbf{s}(Y) \equiv \bigcup_{y \in Y} \{\mathbf{s}(y)\}$ and $\mathbf{t}(Y) \equiv \bigcup_{y \in Y} \{\mathbf{t}(y)\}$. For $Y \subseteq X$, we denote $Y_i \equiv \{y \in Y \mid \mathbf{i}(y) = i\}$; analogously, we denote $Y_s \equiv \{y \in Y \mid \mathbf{s}(y) = s\}$ and $Y_t \equiv \{y \in Y \mid \mathbf{t}(y) = t\}$.

Each student $i \in I$ has a (linear) preference order $P^i$ over contracts in $X_i = \{x \in X \mid \mathbf{i}(x) = i\}$ and an outside option $\emptyset$, which represents remaining unmatched. A contract $x \in X_i$ is acceptable for $i$ (with respect to $P^i$) if $x P^i \emptyset$. We use the convention that $\emptyset, P^i x$ if $x \in X \setminus X_i$. We say that the contracts $x \in X$ for which $\emptyset, P^i x$ are unacceptable to $i$. The at-least-as-well relation $R^i$ is obtained from $P^i$ as follows: $x R^i x'$ if and only if either $x P^i x'$ or $x = x'$. Let $\mathcal{P}^i$ denote the set of all preferences over $X_i \cup \{\emptyset\}$. A preference profile of students is denoted by $P = (P^{i_1},...,P^{i_n}) \in \times_{i \in I} P^i$. A preference profile of all students except student $i_t$ is denoted by $P_{-i_t} = (P^{i_1},...,P^{i_{t-1}},P^{i_{t+1}},...,P^{i_n}) \in \times_{i \neq i_t} P^i$.

Students have unit demand, that is, they choose at most one contract from a set of contract offers. We assume that students always choose the best available contract, so that the choice $C^i(Y)$ of a student $i \in I$ from contract set $Y \subseteq X$ is defined by $C^i(Y) \equiv \max_{P^i Y}$.

For each school $s \in S$, $\overline{q}_s$ denotes the physical capacity of school $s \in S$. We call $\overline{q} = (\overline{q}_1,...,\overline{q}_m)$ the vector of school capacities. Each school $s \in S$ has a priority order $\pi^s$ which is a linear order over $I \cup \{\emptyset\}$.\textsuperscript{18} Let $\Pi = (\pi^{s_1},...,\pi^{s_m})$ denote the priority

\textsuperscript{17}We use the terms "type" and "privilege" interchangeably.

\textsuperscript{18}This priority order is often determined by performance on an admission exam or by a random lottery or dictated by law. In IITs admissions in India each engineering program ranks students according to test scores. Different engineering programs might have different test score rankings as they use different weighted averages of math, physics, chemistry and biology scores depending on the program. It is important to note that students whose test scores are under a certain threshold are deemed as unacceptable for each program.
profile of schools. For each school $s \in S$, the priority ordering for students who can claim the privilege $t \in T$, denoted by $\pi^s_t$ is obtained from $\pi^s$ as follows:

- for any $i, j \in I$ such that $t \in T_i \cap T_j$, $i\pi^s_t j$ if and only if $i\pi^s j$,
- for any $i, j \in I$ such that $t \in T_i \setminus T_j$, $i\pi^s_t \emptyset i\pi^s j$.

An allocation $Y \subseteq X$ is a set of contracts such that each student appears in at most one contract and no school appears in more contracts than its capacity allows. Let $\mathcal{X}$ denote the set of all allocations. Given a student $i$ and an allocation $Y$, we refer to the pair $(s(x), t(x))$ such that $i(x) = i$ as the assignment of student $i$ under allocation $Y$. We extend student preferences over contracts to preferences over outcomes in the natural way. We say that an allocation $Y \subseteq X$ Pareto dominates allocation $Z \subseteq X$ if $Y_i \geq^R_i Z_i$ for all $i \in I$ and $Y_i \geq^P_i Z_i$ for at least one $i \in I$.

2.1 Dynamic Reserves Choice Functions

Each school $s \in S$ has multi-unit demand, and is endowed with a choice function $C^s(\cdot)$ that describes how $s$ would choose from any offered set of contracts. We assume throughout that for all $Y \subseteq X$ and for all $s \in S$, the choice function $C^s(\cdot)$

1. only selects contracts to which $s$ is a party, i.e., $C^s(Y) \subseteq Y_s$,
2. selects at most one contract with any given student.

For any $Y \subseteq X$ and $s \in S$, we denote $R^s(Y) \equiv Y \setminus C^s(Y)$ the set of contracts that $s$ rejects from $Y$.

We now introduce a model of school choice functions in which each school $s \in S$ has $\lambda_s$ groups of slots and fills its groups of slots according to a precedence sequence \(f_s^s : \{1, \ldots, \lambda_s\} \rightarrow T\). The interpretation of $f_s^s$ is that school $s$ fills the first group of seats with $f_s^s(1)$-type students, the second group of seats with $f_s^s(2)$-type students, and so on. Each school $s \in S$ has a target distribution of its slots across different types $(q_{1s}^t, \ldots, q_{\lambda_s}^t)$ which means that $q_{1s}^t$ many slots to be reserved for privilege $t_1$, $q_{2s}^t$ many slots to be reserved for privilege $t_2$, and so on. To satisfy its target reserve structure school $s$ prefers filling its seats according to initially set capacities for each group of seats $(q_{1s}^t, q_{2s}^t, \ldots, q_{\lambda_s}^t)$ such that $\sum_{j \in (f_s^s)^{-1}(t)} q_j^t = q_t^s$, for all $t \in T$. If the target distribution cannot be achieved because too few students from one or more privileges apply, then school $s$ can express its preference over possible alternative distributions.

\[\emptyset \pi^s_t j\] means student $j$ is unacceptable for privilege $t$ at school $s$.

\[^{20}\text{We take precedence sequences as exogenously given. However, in an important work, Dur et. al. (2018) show, in the context of Boston’s school choice system, that precedence sequences might have significant effects on distributional objectives.}\]
by specifying how its capacity is to be redistributed. We consider exogenous capacity transfers that each school announces prior to the allocation of their seats so that it is common knowledge. Technically, a capacity transfer scheme is defined as follows:

**Definition 1.** Given a precedence sequence $f^s$ and a capacity of the first group of seats $q^1_s$, a **capacity transfer scheme** of school $s$ is a sequence of capacity functions $q_s = (q^k_s)_{k=2}^{\lambda_s}$, where $q^k_s : \mathbb{Z}^{k-1}_+ \to \mathbb{Z}_+$ for all $k \in \{2, ..., K_s\}$.

We impose a mild condition on capacity transfer schemes. This condition, **monotonicity**, was first introduced by Westkamp (2013).

**Definition 2.** A capacity transfer scheme $q_s$ is **monotonic** if, for all $j \in \{2, ..., \lambda_s\}$ and all pairs of sequences $(r_l, \tilde{r}_l)$ such that $\tilde{r}_l \geq r_l$ for all $l \leq j - 1$, $q^j_s(r_1, ..., r_{j-1}) \geq q^j_s(\tilde{r}_1, ..., \tilde{r}_{j-1})$, and $\sum_{m=1}^{j-1} [q^m_s(\tilde{r}_1, ..., \tilde{r}_{m-1}) - q^m_s(r_1, ..., r_{m-1})] \leq \sum_{m=1}^{j-1} [\tilde{r}_m - r_m]$.

Monotonicity of capacity transfer schemes requires that (1) whenever weakly more seats are left unfilled in every groups of seats preceding the $j^{th}$ group of seats, weakly more slots should be available for the $j^{th}$ group, and (2) a school cannot decrease total capacity in response to increased demand in some groups of seats.

Note that for each group of seats $j \in \{2, ..., \lambda_s\}$ we have $q^j_s(0, ..., 0) = \overline{q}^j_s$.

**Sub-choice functions**

For each group of slots at school $s \in S$ there is an associated **sub-choice function** $c^s : 2^X \times \mathbb{Z}_+ \times T \to 2^X$. Given a set of contracts $Y \subseteq X$, a nonnegative integer $\kappa \in \mathbb{Z}_+$ and a privilege $t \in T$, $c^s(Y, \kappa, t)$ denotes the set of chosen contracts that name privilege $t$ up to the capacity $\kappa$ from the set of contracts $Y$. We require sub-choice functions to be $q$-responsive given the ranking $\pi^s_t$.

**Definition 3.** A sub-choice function $c^s(\cdot, \kappa, t)$ of a group of seats at school $s$ for privilege type $t$ is **$q$-responsive** if there exists a strict priority ordering $\pi^s_t$ on the set of contracts naming privilege type $t$, and a positive integer $\kappa$ such that for any $Y \subseteq (X_s \cap X_t)$,

$$c^s(Y, \kappa, t) = \bigcup_{i=1}^{\kappa} \{y^*_i\}$$

where $y^*_i$ is defined as $y^*_i = \max_{\pi^s_t} Y$ and, for $2 \leq i \leq \kappa$, $y^*_i = \max_{\pi^s_t} Y \setminus \{y^*_1, ..., y^*_{i-1}\}$.

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We adapt this definition from Chambers and Yenmez (2017).
In words, a sub-choice function is q-responsive\(^{22}\) if there is a strict priority ordering over students who have privilege \(t\) for which the sub-choice function always selects the highest-ranked available students in privilege \(t\) up to the capacity.

\textit{Remark 1.} In the main text, since our main real-life application is the IITs admissions in India, we shall assume that at each school \(s \in S\), and for each group of slots reserved for privilege \(t \in T\) the associated sub-choice function \(c^s(\cdot, \cdot, t)\) is q-responsive and obtained from \(\pi^s_t\).

**Overall choice functions**

The \textit{overall choice function} of school \(s\), \(C^s(\cdot, f^s, q^s) : 2^X \rightarrow 2^X\), runs its sub-choice functions in an orderly fashion given the precedence sequence \(f^s\) and capacity transfer scheme \(q^s\). Given a set of contracts \(Y \subseteq X\), \(C^s(Y, f^s, q^s)\) denotes the set of chosen contracts from the set of contracts \(Y\) and is determined as follows:

- Given \(\overline{q}^1_s\) and \(Y = Y^0 \subseteq X\), let \(Y_1 \equiv c^1_s(\overline{q}^1_s, f^s(1))\) be the set of chosen contracts with privilege \(f^s(1)\) from \(Y^0\). Let \(r_1 = \overline{q}^1_s - |Y_1|\) be the number of vacant seats. Define \(\overline{Y}_1 \equiv \{y \in Y^0 \mid i(y) \in i(Y_1)\}\) the set of all contracts of students whose contract are chosen by sub-choice function \(c^1_s(\cdot, \overline{q}^1_s, f^s(1))\). If a contract of a student is chosen, then all of the contracts naming that student shall be removed from the set of available contracts for the rest of the procedure. The set of remaining contracts is then \(Y^1 = Y^0 \setminus \overline{Y}_1\).

- In general, let \(Y_k = c^k_s(Y^{k-1}, q^k_s, f^s(k))\) be the set of chosen contracts with privilege \(f^s(k)\) from the set of available contracts \(Y^{k-1}\), where \(q^k_s = q^k_s(r_1, ..., r_{k-1})\) is the dynamic capacity of group of seats \(k\) as a function of the vector of the number of unfilled seats \((r_1, ..., r_{k-1})\). Let \(r_k = q^k_s - |Y_k|\) be the number of vacant seats. Define \(\overline{Y}_k \equiv \{y \in Y^{k-1} \mid i(y) \in i(Y_k)\}\). The set of remaining contracts is then \(Y^k = Y^{k-1} \setminus \overline{Y}_k\).

- Given \(Y = Y^0 \subseteq X\) and the capacity of the first group of seats \(\overline{q}^1_s\), we define the \textit{overall choice function} of school \(s\) as \(C^s(Y, f^s, q^s) = c^1_s(Y^0, q^1_s, f^s(1)) \cup (\bigcup_{k=2}^{\lambda_s} c^k_s(Y^{k-1}, q^k_s(r_1, ..., r_{k-1}), f^s(k)))\).

For each school \(s \in S\), precedence sequence \(f^s\), capacity transfer scheme \(q^s\) and q-responsive sub-choice functions of groups of seats \(\{c^k_s(\cdot, q^k_s(\cdot), f^s(k))\}_{k=1}^{\lambda_s}\) are the primitives of the school choices. Since an overall choice is computed by using these primitives it is not one of the primitives of our model.

\(^{22}\)These types of choice functions are often used in real-life applications. For example, in the IITs admissions in India and the cadet-branch matching processes in USMA and ROTC, each sub-choice function is induced from a strict ranking of students according to test scores.
The list \((I,S,T,X,P,\Pi,(f^s,q_s,\{c^s_k(\cdot),q^s_k(\cdot),f^s(k)\})^\lambda_{s=1})_{s\in S}\) denotes a problem.

### 3 Stability concept

Stability has emerged as the key feature to the success of matching market design. We follow the Gale and Shapley (1962) tradition in focusing on outcomes that are stable. In the matching with contracts framework, Hatfield and Milgrom (2005) define stability as follows: an outcome \(Y \subseteq X\) is \textbf{stable} (or \textbf{group-stable}) if

- \(Y_i \not\subset \emptyset\) for all \(i \in I\),
- \(C^s(Y) = Y_s\) for all \(s \in S\), and
- there does not exist a school \(s \in S\) and a \textbf{blocking set} \(Z \neq C^s(Y)\) such that \(Z_s \subseteq C^s(Y \cup Z)\) and \(Z_i = C^i(Y \cup Z)\) for all \(i \in i(Z)\).

If the first requirement (\textit{individual rationality for students}) fails, then there is a student who prefers to reject a contract that involves her (or equivalently, there is a student who is given an unacceptable contract). In our context, the second condition (\textit{individual rationality for schools}) means "respect for school choices" that we explain further below. If the third condition (\textit{unblockedness}) fails, then there is an alternative set of contracts that a school and students associated with a contract in the alternative set strictly prefers.

A conceptually easier notion of stability is \textbf{pairwise-stability}. An outcome \(Y \subseteq X\) is \textbf{pairwise-stable} if

- \(Y_i \not\subset \emptyset\) for all \(i \in I\)
- \(C^s(Y) = Y_s\) for all \(s \in S\), and
- there does not exist a contract \(x \in X \setminus Y\) such that \(x \in C^s(x)(Y \cup \{x\})\) and \(x \in C^i(x)(Y \cup \{x\})\).

The third condition of pairwise-stability considers a single contract between a student and a school. Group-stability and pairwise stability need not be equivalent in general. However, in our setting, if school choices are dynamic reserves, then group-stability and pairwise-stability are equivalent.

\textbf{Proposition 1.} Suppose that each school has a dynamic reserves choice function. Then, an allocation is pairwise stable if and only if it is group stable.

\textit{Proof.} See Appendix 8.2.
We next explain what stability achieves in the present setting by characterizing it with normative axioms.

We say that, given a problem, an outcome $Y \subseteq X$ is feasible if (1) $Y_i \cap Y \neq \emptyset$ for all $i \in I$, and (2) $C^*(Y) = Y_s$. Condition (1) is the standard individual rationality condition for students. Condition (2), on the other hand, is a form of “feasibility” requirement. Since the overall choice function of each school is computed from a given precedence sequence, capacity transfer scheme and priority orders of groups of seats over students, condition (2) requires that precedence sequence, capacity transfer scheme and priority orders of groups of seats must be respected in each school.

We say that, given a problem, an outcome $Y \subseteq X$ satisfies no-justified-envy with other schools if there is no contract $x = (i, s, t) \in X \setminus Y$ such that (1) $x \notin Y_i \cap Y_s$, (2) $x P^i Y_i$ and (3) for all $y \in Y_s \cap Y_i$, we have $i(y) Y_i$. It tells that if student $i$ (whose match under $Y$ is not school $s$) wants to be matched with school $s$ under type $t$, then all of the students who are matched to school $s$ under type $t$ in $Y$ must have higher priority than student $i$ for school $s$ in type $t$. Note that the no-justified-envy with other schools property is independent of the capacity transfer schemes.

We say that, given a problem, an outcome $Y \subseteq X$ satisfies no-justified-envy with own schools if there is no contract $x = (i, s, t) \in X \setminus Y$ such that (1) $x \in Y_i \cap Y_s$, (2) $x P^i Y_i$ and (3) for all $l < k$ where $Y_i$ is chosen by the $k^{th}$ group of seats of the school $s$ given the offer set $Y_s$ such that $f^*(l) = t$, we have $i(y) \pi^i_l i$ for all $y \in Y_s \cap Y_i$ that are chosen by the group of seats $l$. The interpretation of this condition is as follows: if some student $i$ is matched with school $s$ under type $t$ but prefers another contract with the same school under a different type, say $t$, then all of the students who are accepted to school $s$ under type $t$ in groups of seats that precede the group of seats that selected the contract $Y_i$, call it $k$, must have higher priority than her. For the groups of seats that accepts students in type $t$ and follow the group of seats $k$ according to the precedence sequence, priority of student $i$ might be violated. This property is crucial in the present setting because, given precedence sequences, each school tries to stay

\[\text{feasible} = \{0, 1\} \text{ such that } \phi(w) \geq \phi(w') \text{ whenever } w \leq w'. \]

Their interpretation is that each coordinate in $w$ corresponds to a hospital and the number in that coordinate represents the number of doctors matched to that hospital. $\phi(w) = 1$ means that $w$ is feasible and $\phi(w) = 0$ means it is not. They say that matching $\mu$ is feasible if and only if $\phi(w(\mu)) = 1$, where $w(\mu) := (| \mu_h |)_{h \in H}$ is a vector of nonnegative integers indexed by hospitals whose coordinate corresponding to $h$ is $| \mu_h |$. Capacity transfer functions in our setting can be represented by Kamada and Kojima (2017)-style feasibility constraint map. In our definition of feasibility, condition (2) takes into account not only dynamic capacities of groups of seats in each school but also their precedence sequences. Westkamp (2013) provides a similar condition in his “procedural stability” definition in a simpler matching model without contracts.

In the case of IITs admissions in India, condition (2) of the feasibility requirement can be thought as obeying the government’s admission policy as precedence sequences and capacity transfer schemes of schools as well as their priority orders over students are regulated by law.

This type of priority violations is the spirit of the affirmative action.
as close as possible to the target distribution since more students accepted under the types that are intended for them, the less capacity redistribution.

We say that, given a problem, an outcome \( Y \subseteq X \) is **nonwasteful up to the dynamic capacities** if there is no contract \( x = (i, s, t) \in X \setminus Y \) such that either (1) \( x \notin Y_i \cap Y_s \), (2) \( x \mathcal{P} Y_i \) and (3) there exists \( k \in \{1, \ldots, \lambda_s\} \) such that \( f^*(k) = t \) and \( r_k > 0 \) given the offer set \( Y_s \) or (1') \( x \in Y_i \cap Y_s \), (2') \( x \mathcal{P} Y_i \) and (3') for some \( l < k \) where \( Y_i \) is chosen by the \( k^{th} \) group of seats of the school \( s \) given the offer set \( Y_s \) such that \( f^*(l) = t \) we have \( r_l > 0 \) given the offer set \( Y_s \). This condition is an adaptation of the standard non-wastefulness condition to the case with capacity transfers. The first part of this condition ensures that when some student \( i \) prefers some other school \( s \) under type \( t \) to her assignment, then the dynamic capacities of the groups of seats in school \( s \) accepting students in type \( t \) must be exhausted. The second part of the condition states that if some student \( i \) prefers the same school \( s \) under type \( t \) to her assignment, the dynamic capacities of all of the groups of seats that accept students under type \( t \) and precede the group of seats that student \( i \)'s original contract is selected must be exhausted.

With these concepts, we are now ready to offer a characterization of our stability notion.

**Proposition 2.** An outcome \( Y \subseteq X \) is stable if and only if it is feasible and non-wasteful up to the dynamic capacities and satisfies no-justified envy with own and other schools.

*Proof.* See Appendix 8.2. \( \square \)

## 4 Properties of Dynamic Reserves Choice Functions

We now discuss the extent to which dynamic reserves choice functions satisfy the conditions that have been key to previous analyses of matching with contracts models.

**Definition 4.** A choice function \( C^*(\cdot) \) satisfies the **irrelevance of rejected contracts** (IRC) condition if for all \( Y \subset X \), for all \( z \in X \setminus Y \), \( z \notin C^*(Y \cup \{z\}) \implies C^*(Y) = C^*(Y \cup \{z\}) \).

Aygün and Sönmez (2013) show that the IRC condition is necessary for the existence of a stable allocation. Note that the q-acceptant sub-choice functions satisfy the IRC condition. If every sub-choice function satisfies the IRC condition then the overall choice function also satisfies the IRC condition, regardless of the capacity transfer scheme used.

**Proposition 3.** Every dynamic reserve choice function satisfies the IRC condition.

*Proof.* See Appendix 8.2. \( \square \)
Hatfield and Milgrom (2005) introduce the substitutability condition, which generalizes the earlier gross substitutes condition of Kelso and Crawford (1982). Hatfield and Milgrom (2005) also show that substitutability is sufficient to guarantee the existence of stable outcomes. However, their analysis implicitly assumes the IRC condition.

**Definition 5.** A choice function $C^*(\cdot)$ satisfies **substitutability** if for all $z, z' \in X$ and $Y \subseteq X$, $z \notin C^*(Y \cup \{z\}) \implies z \notin C^*(Y \cup \{z, z'\})$.

Substitutability is *not* necessary for the guaranteed existence of stable outcomes in settings where agents have unit demand. Hatfield and Kojima (2010) show that the following condition, which is weaker than substitutability, not only suffices for the existence of stable outcomes but also guarantees that there is no conflict of interest among students.

**Definition 6.** A choice function $C^*(\cdot)$ satisfies **unilateral substitutability** (ULS) if $z \notin C^*(Y \cup \{z\}) \implies z \notin C^*(Y \cup \{z, z'\})$ for all $z, z' \in X$ and $Y \subseteq X$ for which $i(z) \notin i(Y)$.

ULS, together with the law of aggregate demand which is defined below, guarantees the existence of a student-optimal stable allocation, and under them the student-proposing deferred-acceptance mechanism is (group) strategy-proof.

**Proposition 4.** There exists a dynamic reserves choice function that fails ULS.

*Proof.* See Example 1 in Kominers and Sönmez (2016).

Hatfield and Kojima (2010) introduced the following substitutability condition which is weaker than ULS.

**Definition 7.** A choice function $C^*(\cdot)$ satisfies **bilateral substitutability** (BLS) if $z \notin C^*(Y \cup \{z\}) \implies z \notin C^*(Y \cup \{z, z'\})$ for all $z, z' \in X$ and $Y \subseteq X$ for which $i(z), i(z') \notin i(Y)$.

The BLS of a choice function is implied by ULS. The BLS together with the IRC of overall choice functions guarantees the existence of a stable allocation in a many-to-one matching with contracts framework. However, BLS and IRC together are weak conditions in the sense that many well-known properties of stable allocations in the standard matching problem do not carry over to the matching with contracts setting. For instance, the student-optimal stable allocation might fail to exist.

**Proposition 5.** Every dynamic reserves choice function satisfies BLS.

*Proof.* See Appendix 8.2.
Definition 8. A choice function $C^s(\cdot)$ satisfies the **law of aggregate demand** (LAD) if $Y \subseteq Y' \implies |C^s(Y)| \leq |C^s(Y')|$.

That is, the size of the chosen set never shrinks as the set of contracts grows under the LAD. Hatfield and Kojima (2010) show that if choice functions of schools all satisfy US and LAD, every student and school signs the same number of contracts at every stable allocation (i.e., the rural hospital theorem holds). Moreover, the cumulative offer mechanism becomes strategy-proof and weakly Pareto efficient for students.26

**Proposition 6.** There exists a dynamic reserves choice function that fails the LAD.

*Proof.* See Example 2 in Kominers and Sönmez (2016). \qed

**Substitutable Completion of Dynamic Reserves Choice Functions**

The following definitions are due to Hatfield and Kominers (2016). A **completion** of a many-to-one choice function $C^s(\cdot)$ of school $s \in S$ is a choice function $\overline{C}^s(\cdot)$ such that for all $Y \subseteq X$, either $\overline{C}^s(Y) = C^s(Y)$ or there exists a distinct $z, z' \in C^s(Y)$ such that $i(z) = i(z')$. If a choice function $C^s(\cdot)$ has a completion that satisfies the substitutability and IRC condition, then we say that $C^s(\cdot)$ is **substitutably completable**. If every choice function in a profile of choice functions $C = (C^s(\cdot))_{s \in S}$ is substitutably completable, then we say that $C$ is **substitutably completable**.

Let $C^s(\cdot, f^s, q_s)$ be a dynamic reserve choice function given the precedence sequence $f^s$ and the capacity transfer scheme $q_s$. We define a related choice function $\overline{C}^s(\cdot, f^s, q_s)$. Given a set of contracts $Y \subseteq X$, $\overline{C}^s(Y, f^s, q_s)$ denotes the set of chosen contracts from the set of contracts $Y$ and is determined as follows:

- Given $q_s^1$ and $Y = Y^0 \subseteq X$, let $Y_1 = c_1^s(Y^0, q_s^1, f^s(1))$ be the set of chosen contracts with privilege $f^s(1)$ from $Y^0$. Let $r_1 = q_s^1 - |Y_1|$ be the number of vacant seats. The set of remaining contracts is then $Y^1 = Y^0 \setminus Y_1$.

- In general, let $Y_k = c_k^s(Y^{k-1}, q_s^k, f^s(k))$ be the set of chosen contracts with privilege $f^s(k)$ from the set of available contracts $Y^{k-1}$, where $q_s^k = q_s^k(r_1, ..., r_{k-1})$ is the dynamic capacity of group of seats $k$ as a function of the vector of the number of unfilled seats $(r_1, ..., r_{k-1})$. Let $r_k = q_s^k - |Y_k|$ be the number of vacant seats. The set of remaining contracts is then $Y^k = Y^{k-1} \setminus Y_{k-1}$.

- Given $Y = Y^0 \subseteq X$ and the capacity of the first group of seats $q_s^1$, we define $\overline{C}^s(Y, f^s, q_s) = c_1^s(Y^0, q_s^1, f^s(1)) \cup \bigcup_{k=2}^{\lambda_s} c_k^s(Y^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(k)))$.

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26If schools do not have preferences that generate their choices, then all of these results are obtained under the additional IRC condition of Aygün and Sönmez (2012).
The difference between $C^s(\cdot)$ and $\overline{C}^s(\cdot)$ is the following: In the computation of $C^s(\cdot)$ if a contract of a student is chosen by some group of seats then his/her other contracts are removed for the rest of the choice procedure. However, in the computation of $\overline{C}^s(\cdot)$ this is not the case. According to the choice procedure $\overline{C}^s(\cdot)$ if a contract of a student is chosen, say, group of seats $k$, then his/her other contracts will still be available for the following groups of seats.

The following proposition shows that $\overline{C}^s(\cdot)$ defined above is the completion of the dynamic reserves choice function $C^s(\cdot)$.

**Proposition 7.** $\overline{C}^s(\cdot)$ is a completion of $C^s(\cdot)$.

*Proof.* See Appendix 8.2.

We will first show that for any dynamic reserves choice function $C^s(\cdot)$ its completion $\overline{C}^s(\cdot)$ satisfies the IRC condition.

**Proposition 8.** $\overline{C}^s(\cdot)$ satisfies the IRC.

*Proof.* See Appendix 8.2.

The proof of Proposition 8 relies on the sub-choice functions satisfying the IRC condition. The completion $\overline{C}^s(\cdot)$ also satisfies the substitutes condition.

**Proposition 9.** $\overline{C}^s(\cdot)$ satisfies the substitutability.

*Proof.* See Appendix 8.2.

The last proposition in this section states that the completion defined satisfies the LAD. The proofs of Propositions 8 and 9 rely on the q-responsiveness of the sub-choice functions and monotonicity of the capacity transfer schemes.

**Proposition 10.** $\overline{C}^s(\cdot)$ satisfies the LAD.

*Proof.* See Appendix 8.2.

The completion $\overline{C}^s(\cdot)$ of the dynamic reserves choice function $C^s(\cdot)$, given the precedence sequence $f^s$ and monotonic capacity transfer scheme $q_s$, satisfying the IRC, substitutability and LAD enable us to utilize the theory developed by Hatfield and Kominers (2016) in important real life applications in which it is natural for institutions to transfer otherwise vacant slots from low-demand groups to high-demand groups.
Dynamic Reserves Choice Functions vs. Slot Specific Priorities Choice Functions

Slot specific priorities choice functions of Kominers and Sönmez (2016) are another prominent family of choice functions in order to implement diversity objectives. The richness of these types of choice functions comes from the fact that each slot might have its own independent priority over contracts. This section analyzes the relationship between the dynamic reserves choice functions and slot specific priorities choice functions. Even though the intersection of these two large families of choice functions is non-empty, each family has choice functions that cannot be generated in the other family. We provide two such choice function examples where the first one can be represented by dynamic reserves but not with slot specific priorities whereas the second one can be represented by slot specific priorities but not with dynamic reserves.

Example 1. A dynamic reserve choice function that cannot be generated by slot specific priorities.

Consider the following problem with \( I = \{i, j, k, l\} \) and \( S = \{s\} \) with \( q_s = 2 \). There are three different types of students, i.e., \( \Theta = \{t_1, t_2, t_3\} \). Student \( i \) only has type \( t_1 \) and hence a single contract \( x_1 \). Student \( j \) only has type \( t_2 \) and a single contract \( y_2 \). Student \( k \) has two types: type \( t_2 \) and type \( t_3 \); and two contracts related to these types \( z_2 \) and \( z_3 \), respectively. Finally, student \( l \) has two types: type \( t_1 \) and type \( t_3 \); and two contracts related to these types \( w_1 \) and \( w_3 \). Hence, the set of contracts for this problem is \( X = \{x_1, y_2, z_2, z_3, w_1, w_3\} \). Students have the following test score ordering from highest to lowest: \( i - j - k - l \).

The school reserves the first seat for type \( t_1 \) students, and the second seat for type \( t_2 \) students. If either the first seat or the second seat cannot be filled with students whom the seats are reserved for, they are filled with type \( t_3 \) student(s). The precedence order is such that first seat is filled first with a type \( t_1 \) student if possible, and then the second seat is filled with type \( t_2 \) student if possible. If any of these seats cannot be filled with the intended student types, all of the vacant seats are filled with type \( t_3 \) student(s) at the very end, if possible.

We can represent the distributional objective described above by capacity-transfers as follows: Initially \( q_{t_1} = q_{t_2} = 1 \) and \( q_{t_3} = 0 \). If either of the first two seats cannot be filled, \( q_{t_3} = r_1 + r_2 \) where \( r_1, r_2 \in \{0, 1\} \). Some of the choice situations under the capacity-transfer described above are given below:
In order to implement the choices above with slot-specific priorities, we need to find a strict ranking of the contracts in $X$ for both of the slots. Since \{x_1, y_2\} is chosen from the grand set, from one of the slots $x_1$ and from the other slot $y_2$ must be chosen.

Case 1: $x_1$ is chosen from slot 1 and $y_2$ is chosen from slot 2. Then, $x_1$ has the highest priority in slot 1. Since $C(\{x_1, z_2, z_3\}) = \{x_1, z_2\}$, then $z_2$ must have higher priority than $z_3$ in the strict priority ranking of slot 2 because $x_1$ will be chosen from the first slot. Notice that both $z_2$ and $z_3$ must have lower priority than $y_2$ in the strict ranking of slot 2. Also, since $C(\{y_2, z_2, z_3\}) = \{y_2, z_3\}$, then it must be the case that $z_3$ has higher priority than $z_2$ in the strict priority of the first slot. Notice that $z_3$ cannot be chosen from the second slot as $z_2$ has higher priority. However, $C(\{z_2, z_3\}) = \{z_2\}$. Contradiction.

Case 2: $y_2$ is chosen from slot 1 and $x_1$ is chosen from slot 2. Then, $y_2$ has the highest priority in slot 1. Since $C(\{y_2, w_1, w_3\}) = \{y_2, w_1\}$, then in the ranking of slot 2 $w_1$ must have higher priority than $w_3$. Also, since $C(\{x_1, w_1, w_3\}) = \{x_1, w_3\}$, it follows that in the ranking of slot 1 $w_3$ must have higher priority than $w_1$ because $w_3$ cannot be chosen from slot 2 as it has a lower priority than $w_1$ there. However, $C(\{w_1, w_3\}) = \{w_1\}$. Contradiction.

Hence, the distributional objective described above can be achieved by capacity-transfers and cannot be implemented by slot-specific priorities.

**Example 2.** A slot specific priorities choice function that cannot be generated by dynamic reserves.

Consider the following preference relation for a school $s$:

$$\{x_1, y_2\} \succ_s \{x_1, x_2\} \succ_s \{x_1\} \succ_s \{x_2\} \succ_s \{y_2\} \succ_s \emptyset.$$

Suppose that $i(x_1) = i$ and $i(x_2) = i(y_2) = j$. The choice function coming from this preference relation can be generated as a slot specific priorities choice function

\[\text{Contradiction.}\]
as follows: There are two slots of school $s$. The first slot $s^1$ has the priority ranking $x_1 \succ_s x_2 \succ_s y_2$ and the second slot $s^2$ has the priority ranking $y_2 \succ_s x_2 \succ_s x_1$.

Since the contracts $x_2$ and $y_2$ are with the same agent, they must represent contracts for two different privilege types, which means no sub-choice function can consider them at the same time according to the formulation of dynamic reserves choices. But from the choice set $\{x_2, y_2\}$ it must be that contract $x_2$ is chosen, which means that the sub-choice function from the type $t(x_2)$ is considered before the type $t(y_2)$. However, from the choice set $\{x_1, x_2, y_2\}$ it must be that $\{x_1, y_2\}$ is chosen. Therefore, $x_1$ must prevent $x_2$ from being chosen. There are two cases:

Case 1: $x_1$ and $x_2$ are of the same type, in which case this type, which must be considered first in the precedence sequence by the argument, has a target capacity of 1.

Case 2: $x_1$ and $x_2$ are of different types, in which case $x_2$ being chosen when $x_1$ is not present implies that the type $t(x_1)$ is considered before the type $t(x_2)$ which in turn is considered before the type $t(y_2)$, as established before. Moreover, the target capacity of the type $t(x_2)$ must be zero for $x_2$ not be chosen from the choice set $\{x_1, x_2, y_2\}$, but receives a capacity transfer from the type $t(x_1)$ when that type’s capacity is not filled, which explains why $x_2$ is chosen when $x_1$ is not present.

However, both cases lead to the conclusion that from choice set $\{x_1, x_2\}$ only $x_1$ is chosen, rather than both contracts.

The limitation of dynamic reserves functions is that each q-responsive sub-choice function uses type-specific priority ordering that is obtained from the school’s priority ordering over students and no two contract of the same student can be considered by the same sub-choice function as explained in Example 2. The limitation of slot specific priorities choice functions, on the other hand, is that it does not allow transferring capacity if a slot remains unfilled.

5 The Cumulative Offer Mechanism and its Properties under Dynamic Reserves Choice Functions

A direct mechanism is a mechanism where the strategy space is the set of preferences $P$ for each student $i \in I$, i.e., a function $\psi : P^n \rightarrow X$ that selects an allocation for each preference profile. We propose the cumulative offer mechanism (COM) as our allocation function. Given the student preferences and schools’ overall choice functions the outcome of the COM is computed by the cumulative offer algorithm, which is the generalization of the agent-proposing deferred acceptance algorithm of Gale and
Shapley (1962). We now introduce the cumulative offer process\(^{28}\) for matching with contracts. Here, we provide an intuitive description of this algorithm; we give a more technical description in Appendix 8.1.

Definition 9. In the cumulative offer process, students propose contracts to schools in a sequence of steps \(l = 1, 2 \ldots \):

**Step 1**: Some student \(i^1 \in I\) proposes his most-preferred contract, \(x^1 \in X_{i^1}\). School \(s(x^1)\) holds \(x^1\) if \(x^1 \in C^s(x^1)\), and rejects \(x^1\) otherwise. Set \(A^2_s(x^1) = \{x^1\}\), and set \(A^2_s = \emptyset\) for each \(s' \neq s(x^1)\); these are the sets of contracts available to schools at the beginning of Step 2.

**Step 2**: Some student \(i^2 \in I\), for whom no contact is currently held by any school, proposes his most-preferred contract that has not yet been rejected, \(x^2 \in X_{i^2}\). School \(s(x^2)\) holds the contract in \(C^s(x^2)\) \(\left( A^2_s(x^2) \cup \{x^2\} \right) \) and rejects all other contracts in \(A^2_s(x^2) \cup \{x^2\}\); schools \(s' \neq s(x^2)\) continue to hold all contracts they held at the end of Step 1. Set \(A^3_s(x^2) = A^2_s(x^2) \cup \{x^2\}\), and set \(A^3_s = A^2_s\) for each \(s' \neq s(x^2)\).

**Step \(l\)**: Some student \(i^l \in I\), for whom no contact is currently held by any school, proposes his most-preferred contract that has not yet been rejected, \(x^l \in X_{i^l}\). School \(s(x^l)\) holds the contract in \(C^s(x^l)\) \(\left( A^l_s(x^l) \cup \{x^l\} \right) \) and rejects all other contracts in \(A^l_s(x^l) \cup \{x^l\}\); schools \(s' \neq s(x^l)\) continue to hold all contracts they held at the end of Step \(l-1\). Set \(A^{l+1}_s(x^l) = A^l_s(x^l) \cup \{x^l\}\), and set \(A^{l+1}_s = A^l_s\) for each \(s' \neq s(x^l)\).

If at any time no student is able to propose a new contract, that is, if all students for whom no contracts are on hold have proposed all contract they find acceptable, then the algorithm terminates. The **outcome of the cumulative offer process** is the set of contracts held by schools at the end of the last step before termination.

In the cumulative offer process, students propose contracts sequentially. Schools accumulate offers, choosing at each step (according to \(C^s\)) a set of contracts to hold from the set of all previous offers. The process terminates when no student wishes to propose a contract.

Given a preference profile of students \(P = (P_i)_{i \in I}\) and a profile of choice functions for schools \(C = (C^s)_{s \in S}\), let \(\Phi (P, C)\) denote the outcome of the COM. Let \(\Phi_s (P, C)\) denote the assignment of student \(i \in I\) and \(\Phi_s (P, C)\) denote the assignment of school \(s \in S\).

Remark 2. We do not explicitly specify the order in which students make proposals. Hirata and Kasuya (2014) show that in the matching with contracts model, the outcome of the cumulative offer process is order-independent if the overall choice function of every school satisfies the bilateral substitututability and the irrelevance of rejected contracts.

\(^{28}\)See Hatfield and Milgrom (2005) for more details.
conditions. In our setup, the dynamic reserves choice function of every school satisfies BLS and IRC, and hence, the order-independence of the cumulative offer holds.

A mechanism \( \phi \) is stable if for every preference profile \( P \in \mathcal{P}^{|I|} \) the outcome \( \phi(P) \) is stable with respect to schools’ overall choice functions. The COM is a stable mechanism since the cumulative offer process gives a stable outcome for every input if each school’s capacity transfer scheme is monotonic.

**Theorem 1.** The cumulative offer mechanism, \( \Phi \), is stable with respect to dynamic reserves choice functions.

**Proof.** See Appendix 8.2. \( \square \)

If the overall choice functions of schools satisfy ULS and LAD, then the COM is (group) strategy-proof.\(^{29}\) Even though ULS and LAD are sufficient for strategy-proofness, they are not necessary.\(^{30}\) Some choice functions in the family of slot specific priorities of Kominers and Sönmez (2016) might violate both ULS and LAD. However, they show that the COM is strategy-proof when each school uses a slot-specific priorities choice function. Later, Hatfield and Kominers (2016) show that if schools’ choice functions have substitutable completions so that these completions satisfy the LAD, then the cumulative offer process becomes strategy-proof under these choice functions.

To analyze the incentive properties of the COM when schools use dynamic reserves choice functions we first define standard strategy-proofness and (weak) group strategy-proofness properties of a mechanism.

**Definition 10.** A direct mechanism \( \phi \) is said to be **strategy-proof** if there does not exist a preference profile \( P \), a student \( i \in I \) and preferences \( P_i' \) of student \( i \) such that

\[
\phi_i(P_i', P_{-i}) P_i \phi_i(P).
\]

That is, no matter which student we consider, no matter what her true preferences \( P_i \) are, no matter what other preferences \( P_{-i} \) other students report (true or not), and no matter which potential “misrepresentation” \( P_i' \) student \( i \) considers, truthful preference revelation is in her best interests. Hence, students can never benefit from “gaming” the mechanism \( \phi \).

**Definition 11.** A direct mechanism \( \phi \) is said to be **weak group strategy-proof** if there is no preference profile \( P \), a subset of students \( I' \subseteq I \), and a preference profile

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\(^{29}\)See Hatfield and Kojima (2010) and Aygün and Sönmez (2012).

\(^{30}\)See Hatfield and Kominers (2016) and Hatfield, Kominers and Westkamp (2017b) for more on this issue.
\((P_i)_{i \in I'}\) of students in \(I'\) such that

\[
\varphi_i\left((P'_i)_{i \in I'}, (P_j)_{j \in I\setminus I'}\right) R_i \varphi_i(P)
\]

for all \(i \in I'\).

That is, no subset of students can jointly misreport their preferences to receive a strictly preferred outcome for every member of the coalition under the mechanism.

Our proof for group strategy-proofness of the cumulative offer mechanism with respect to the dynamic reserves choice functions we developed is based on the following result from Hatfield and Kominers (2016).

**Theorem 2.** (Theorem 3 of Hatfield and Kominers, 2016) Let \(\varphi(C^I, C^S)\) be the mechanism that implements the student-proposing deferred-acceptance algorithm given choice profile \((C^I, C^S)\). If, for each \(s \in S\), the choice function \(C^s\) has a substitutable completion that satisfies the law of aggregate demand, then \(\varphi(C^I, C^D)\) is group strategy-proof for students.

**Theorem 3.** Suppose that each school uses a dynamic reserves choice function. Then, the cumulative offer mechanism is weak group strategy-proof.

**Proof.** See Appendix 8.2.

## Respect for Unambiguous Improvements

We say that priority profile \(\Pi\) is an unambiguous improvement over priority profile \(\Pi\) for student \(i \in I\) if for all schools \(s \in S\), the following conditions hold:

1. For all \(x \in X_i\) and \(y \in (X_{I\setminus \{i\}} \cup \{\emptyset\})\), if \(x \pi^s y\) then \(x \bar{\pi}^s y\).

2. For all \(y, z \in X_{I\setminus \{i\}}\), \(y \bar{\pi}^s z\) if and only if \(y \pi^s z\).

That is, \(\Pi\) is an unambiguous improvement over priority profile \(\Pi\) for student \(i\) if \(\Pi\) is obtained from \(\Pi\) by increasing the priority of some of \(i\)'s contracts at some groups of seats at some schools while leaving the relative priority of other students’ contracts unchanged.

**Definition 12.** A mechanism \(\varphi\) respects unambiguous improvements for \(i \in I\) if for any preference profile \(P \in \times_{i \in I} P^i\)

\[
\varphi_i(P; \Pi) R^i \varphi_i(P; \Pi)
\]

whenever \(\Pi\) is an unambiguous improvement over \(\Pi\) for \(i\). We say that \(\varphi\) respects unambiguous improvements if it respects unambiguous improvements for each student \(i \in I\).
Respect for improvements is essential in settings like the admissions to IITs in India where it implies that students never want to intentionally decrease their test scores, and hence, rankings. Similarly, respect for unambiguous improvements is important in cadet-branch matching where cadets can influence their priority rankings directly. Sönmez (2013) argues that cadets take perverse steps to lower their priorities because the mechanism used by the Reserve Officer Training Corps (ROTC) to match its cadets to branches fails the respecting improvements property.

Respect for improvements is also important in the normative literature on school choice. This condition is, for example, used by Hatfield, Kojima and Narita (2015) to study how school choice mechanism selection can influence schools’ incentives for self-improvement. Kominers and Sönmez (2016) show that the cumulative offer mechanism respects unambiguous improvements when accepting side employs slot specific priorities choice functions. Their result generalizes Proposition 4 of Sönmez and Switzer (2013). We provide a similar result for the family of dynamic reserves choice functions.

**Theorem 4.** The cumulative offer mechanism with respect to dynamic reserves choice functions respects unambiguous improvements.

**Proof.** See Appendix 8.2.

6 Comparative Statics on Monotonic Capacity Transfer Schemes

In this section, we first define a comparison criteria between two monotone capacity transfer schemes. Consider a school \( s \in S \) with a given precedence sequence \( f^s \) and target distribution \( \bar{q}_s = (\bar{q}_s^1, ..., \bar{q}_s^{\lambda_s}) \). Let \( q_s \) and \( \bar{q}_s \) be two monotone capacity transfer schemes: given a vector of unused seats from group of seats 1 to \( j - 1 \), i.e., \( (r_1, ..., r_{j-1}) \in \mathbb{Z}_{+}^{j-1} \), the dynamic capacity of the \( j \)th group of seats under capacity transfer schemes \( q_s \) and \( \bar{q}_s \) are \( q_s^j = q_s^j(r_1, ..., r_{j-1}) \) and \( \bar{q}_s^j = \bar{q}_s^j(r_1, ..., r_{j-1}) \), respectively, for all \( j \geq 2 \) and, \( q_s^1 = \bar{q}_s^1 = q_1^s \).

Let \( q_s \) and \( \bar{q}_s \) be two monotone capacity transfer schemes that are compatible with the precedence sequence \( f^s \) and target capacity vector \( \bar{q}_s \) of school \( s \in S \). We say that the monotone capacity transfer scheme \( \bar{q}^a \) is **more flexible** than the monotone capacity transfer scheme \( q^a \) if

1. there exists \( l \in \{2, ..., \lambda_s\} \) and \( (\hat{r}_1, ..., \hat{r}_{l-1}) \in \mathbb{Z}_{+}^{l-1} \) such that \( \bar{q}_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) > q_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) \), and
2. for all \( j \in \{2, ..., \lambda_s\} \) and \( (r_1, ..., r_{j-1}) \in \mathbb{Z}_{+}^{j-1} \), if \( j \neq l \) or \( (r_1, ..., r_{j-1}) \neq (\hat{r}_1, ..., \hat{r}_{l-1}) \), then \( \bar{q}_s^j(r_1, ..., r_{j-1}) \geq q_s^j(r_1, ..., r_{j-1}) \).
The definition states that a monotonic capacity transfer scheme is more flexible than another one if the first one transfers at least as many vacant seats as the second one at every instance, i.e., vectors of the number of unused seats, and there must be an instance where the first one transfers strictly more vacant seats than the second one to the next group of seats according to the precedence sequence. Also, both of the monotonic capacity transfer schemes take the capacity of the first group of seats with respect to the precedence sequence equal to its target capacity.

We are now ready to define a notion of choice function expansion.

**Definition 13.** Consider school $s \in S$ with the precedence sequence $f^s$ and target distribution $q^s$. Let $\tilde{C}^s(\cdot, f^s, q^s)$ and $C^s(\cdot, f^s, q^s)$ be two overall choice functions of school $s$ under monotone capacity transfer schemes $\tilde{q}_s$ and $q_s$, respectively. We say that choice function $\tilde{C}^s(\cdot, f^s, q^s)$ is an expansion of choice function $C^s(\cdot, f^s, q^s)$ if $\tilde{q}_s$ is more flexible than $q_s$.

It is important to note that our notion of choice function expansion is different than the one Chambers and Yenmez (2017) define. Their notion of expansion is in the sense of set inclusion while ours is not. They say that a choice function $C'$ is an expansion of another choice function $C$ if for every offer set $Y$, $C(Y) \subseteq C'(Y)$. According to our notion of expansion, when a choice function $C$ expands to $C'$ it is possible to have $C(Y) \nsubseteq C'(Y)$ for some $Y$.

Expanding schools’ overall choice function by implementing a more flexible capacity transfer scheme never leads to a decrease in the number of chosen contracts for any given offer set.

**Theorem 5.** Suppose that $\tilde{C}^s(\cdot, f^s, \tilde{q}_s)$ is an expansion of $C^s(\cdot, f^s, q_s)$ given precedence sequence $f^s$. Then, for any offer set $Y \subseteq X$, $|\tilde{C}^s(Y, f^s, \tilde{q}_s)| \geq |C^s(Y, f^s, q_s)|$.

**Proof.** See Appendix 8.2.

Moreover, expanding the overall choice function of a single school leads to Pareto improvement for students under the cumulative offer mechanism.

**Theorem 6.** Suppose that $Z$ is the outcome of the cumulative offer mechanism at $(P, C)$ where $P = (P_{11}, \ldots, P_{in})$ is the profile of student preferences and $C = (C^s_1, \ldots, C^s_m)$ is the profile of schools’ overall choice functions. Fix a school $s \in S$. Suppose that $\tilde{C}^s$ is an expansion of $C^s$. Then, the outcome of the cumulative offer mechanism at $(P, (\tilde{C}^s, C_{-s}))$, $\tilde{Z}$, Pareto dominates $Z$.

**Proof.** See Appendix 8.2.
Theorem 6 is of particular importance because it tells that increasing the transferability of capacity from low-demand groups to high-demand groups leads to strategy-proof Pareto improvement with the cumulative algorithm. This result provides a normative foundation for recommending a more flexible interpretation of type-specific quotas. This result establishes that to maximize students’ welfare, schools’ choice functions should be expanded as much as possible.

7 Conclusion

This paper studies a matching problem with distributional objectives where agents care not only about the institution they are matched with but also about the contractual terms of the contract with the institution. In other words, we expand the preference domain of agents from institutions only to institutions-contractual terms pairs. Each institution can be thought of as union of different divisions, where each division is associated with exactly one contractual term. Institutions have target distributions over their divisions in the form of reserves. If these reserves are considered to be hard bounds, then in the case that demand for a particular division is less than its target capacity, some slots will remain empty. To overcome this problem and to increase efficiency, we design a family of choice functions we call dynamic reserves choice function. We do so by allowing monotonic capacity transfers across divisions when one or more of the divisions is not able to fill to its target capacity. The capacity transfer scheme is exogenously given for each institution and governs the dynamic capacities of groups of seats, each of which has a q-responsive sub-choice function. The overall choice function of an institution can be thought of as the union of choices with these sub-choice functions of its groups of seats.

We first show that pairwise and group stability are equivalent for the problem we consider and we characterize the stability notion with normative properties. We analyze the properties of the choice functions in the dynamic reserves choice functions family. We offer the cumulative offer mechanism with respect to dynamic reserves choice functions as an allocation rule. We show, by utilizing the substitutable completion theory of Hatfield and Kominers (2016), that the cumulative offer mechanism is stable and strategy proof in our framework. Moreover, the cumulative offer mechanism respects unambiguous improvement. We introduce a comparison criteria between two monotonic capacity transfer schemes. If a monotone capacity transfer scheme transfers at least as many vacancies in every contingency compared to another monotone capacity transfer scheme, we say that the first one is more flexible than the second one. We show that every choice function in the dynamic reserve choice family selects weakly more contracts under more flexible capacity transfer schemes. Finally, and more importantly,
we show that when capacity transfer scheme of a school becomes more flexible while other schools choice functions remains unchanged the outcome of the cumulative offer mechanism under modified profile of choice functions Pareto dominates the outcome of the cumulative offer mechanism under the original profile of choice functions. The last result is the main message of our paper because it describes a strategy proof Pareto improvement by making capacity transfers more flexible.

8 Appendices

8.1 Formal Description of the Cumulative Offer Process and Deferred Acceptance Algorithm

**Cumulative Offer Process:** For any preference profile $P$ of students, profile of choice functions of schools $C$ and an ordering $\Gamma$ of the elements of $X$, the outcome of the cumulative offer mechanism, denoted by $\Phi^\Gamma(P,C)$, is determined by the cumulative offer process with respect to $\Gamma$, $P$ and $C$ as follows:

**Step 0:** Initialize the set of contracts available to the schools as $A_0 = \emptyset$.

**Step $t \geq 1$:** Consider the set

$$U_t \equiv \{ x \in X \setminus A_{t-1} : i(x) \notin i(C^S(A_{t-1})) \text{ and } \exists z \in (X_{i(x)} \setminus A_{t-1}) \cup \emptyset \text{ such that } zP^i(x) \}.$$

If $U_t$ is empty, then the algorithm terminate and the outcome is given by $C^S(A_{t-1})$.

Otherwise, letting $y_t$ be the highest-ranked element of $U_t$ according to $\Gamma$, we say that $y_t$ is *proposed* and set $A_t = A_{t-1} \cup \{ y_t \}$ and proceed to step $t+1$.

A cumulative offer process begins with no contracts available to the schools (i.e., $A^0 = \emptyset$). Then, at each step $t$, we construct $U_t$, the set of contracts that (1) have not yet been proposed, (2) are not associated to students with contracts chosen by schools from the currently available set of contracts, and (3) are both acceptable and the most-preferred by their associated students among all contracts not yet proposed. If $U_t$ is empty, then every student $i$ either has some associated contract chosen by some school, i.e., $i \in i(C^S(A_{t-1}))$, or has no acceptable contracts left to propose, and so the cumulative offer process ends. Otherwise, the contract in $U_t$ that is highest-ranked according to $\Gamma$ is proposed by its associated student, and the process proceeds to the

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31 We borrowed the descriptions of the cumulative offer process with respect to $\Gamma$, $P$ and $C$ and its related discussions from Hatfield, Kominers and Westkamp (2017b).

32 We denote by $C^S(Y) \equiv \bigcup_{s \in S} C^s(Y)$ the set of contracts chosen by the set of schools from a set of contracts $Y \subseteq X$. 
next step. Note that at some step this process must end as the number of contracts is finite.

Letting $T$ denote the last step of the cumulative offer process with respect to $\Gamma$, $P$ and $C$, we call $A^T$ the set of contracts observed in the cumulative offer process with respect to $\Gamma$, $P$ and $C$.

8.2 Proofs

Proof of Proposition 1  It is straightforward, by definition, to see that group-stability implies pairwise-stability. We need to show that pairwise stability implies group stability when school choices are dynamic reserves choices. Let $Y$ be a pairwise stable outcome. Towards a contradiction, suppose that it is not group-stable. Then, there must exists a set of contracts $Z = \bigcap_1^k \bigcup_{s}^C \bigcup_{q}^f (Y \cup Z, f^k, q^f(k)) = \emptyset$ and for every $k < k^*$ $Z \cap c_k^p ((Y \cup Z)^{k-1}, q^k, f^p(k)) = \emptyset$. In words, $k^*$ is the first group of seats in the precedence sequence of school $s$ in which a contract from $Z$ is chosen by its sub-choice function. By definition of $k^*$, $c_k^p ((Y \cup Z)^{k-1}, q^k, f^p(k))$ for all $k < k^*$ since each sub-choice function is q-responsive and hence satisfies the IRC condition. Note that when school $s$ faces choice situations $Y$ and $Y \cup Z$, the dynamic capacity of the group of seats $k^*$ are the same under both since its dynamic capacity is a function of the number of vacant seats in group of seats $1, ..., k^* - 1$. Let $z \in Z \cap c_k^p ((Y \cup Z, q^k, f^p(k)))$, i.e., $z$ is one of the contracts from $Z$ that is chosen by the group of seats $k^*$ in the choice situation $Y \cup Z$. Now consider two choice situations: $Y$ and $Y \cup \{z\}$. Since each sub-choice function is q-responsive (and hence satisfies the IRC condition) for all $k < k^*$ we have $c_k^p ((Y \cup \{z\})^{k-1}, q^k, f^p(k))$. Hence, under choice situations $Y$ and $Y \cup \{z\}$ the dynamic capacities of the group of seats $k^*$ are the same. Given $z \in Z \cap c_k^p ((Y \cup \{z\})^{k^*-1}, q^*_{k^*}, f^p(k^*))$ and q-responsiveness of the sub-choice functions (note that q-responsive sub-choice function satisfy substitutability condition) $z \in c_k^p ((Y \cup \{z\})^{k^*-1}, q^*_{k^*}, f^p(k^*))$. Hence, the contract $z$ is chosen by the overall choice function of school $s$ from $Y \cup \{z\}$. This means that $\{z\}$ blocks $Y$ which was assumed to be pairwise stable. Hence, a contradiction.

Proof of Proposition 2

The “only if” part  Suppose that $Y$ is a stable outcome. Then, by definition of stability, $Y$ is feasible.

To show that it satisfies no-justified-envy with other schools assume, towards a contradiction, that $Y$ does not satisfy no-justified-envy with other schools. Then
there exists a contract \( x = (i, s, t) \in X \setminus Y \) such that (1) \( x \notin Y_i \cap Y_s \), (2) \( xP^iY_i \) and (3) for some \( y \in Y_s \cap Y_t \) such that \( i \pi_1(y) \). Let \( k \in \{1, ..., \lambda_s\} \) such that \( y \in c_s^k(Y_s^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(k)) \). If there exists \( l < k \) such that \( f^s(l) = f^s(k) \) and \( x \in c_s^l((Y_s \cup \{x\})^{l-1}, q_s^l(r_1, ..., r_{l-1}), f^s(l)) \), then we have a contradiction that \( Y \) is a stable outcome. Otherwise, suppose that \( x \) is not chosen by any group of seats \( l < k \). The dynamic capacity of group of seats under offer sets \( Y_s \) and \( Y_s \cup \{x\} \) are the same. Since sub-choice functions are q-responsive \( y \in c_s^k(Y_s^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(k)) \) implies \( x \in c_s^k((Y_s \cup \{x\})^{l-1}, q_s^l(r_1, ..., r_{l-1}), f^s(l)) \). Then, \( x \in C^s(Y \cup \{x\}, f^s, q_s) \) implying that \( Y \) is not stable.

To show that it satisfies no-justified-envy with own schools assume, towards a contradiction, that \( Y \) does not satisfy no-justified-envy with own schools. Then there exists a contract \( x = (i, s, t) \in X \setminus Y \) such that (1) \( x \notin Y_i \cap Y_s \), (2) \( xP^iY_i \) and (3) for some \( l < k \) where \( Y_i \) is chosen by the \( k \)-th group of seats of the school \( s \) given the offer set \( Y_s \) such that \( f^s(l) = t \), there exists a \( y \in Y_s \cap Y_t \) that are chosen by the group of seats \( l \) such that \( i \pi_1(y) \). Now, consider the offer set \( Y_s \cup \{x\} \). If \( x \) is chosen by a group of seats \( l' \) of school \( s \) that precede the group of seats \( l \) such that \( f^s(l') = f^s(l) \), then we have a contradiction that \( Y \) is a stable outcome. Otherwise, suppose that \( x \) is not chosen before the group seats \( l \). Then, q-responsiveness of the sub-choice function \( c_s^k((Y_s \cup \{x\})^{l-1}, q_s^l(r_1, ..., r_{l-1}), f^s(l)) \) together with the facts that \( y \in c_s^k((Y_s^{l-1}, q_s^l(r_1, ..., r_{l-1}), f^s(l)) \) and \( i \pi_1(y) \) implies that \( x \in c_s^k((Y_s \cup \{x\})^{l-1}, q_s^l(r_1, ..., r_{l-1}), f^s(l)) \). Then, \( x \in C^s(Y \cup \{x\}, f^s, q_s) \) implying that \( Y \)'s not stable.

To show that \( Y \) is non-wasteful up to the dynamic capacities assume, towards a contradiction, that \( Y \) is not non-wasteful up to the dynamic capacities. Then there exists a contract \( x = (i, s, t) \in X \setminus Y \) such that either (1) \( x \notin Y_i \cap Y_s \), (2) \( xP^iY_i \) and (3) there exists \( l \in \{1, ..., \lambda_s\} \) such that \( f^s(l) = t \) and \( r_l > 0 \) given the offer set \( Y_s \) or (1') \( x \in Y_i \cap Y_s \), (2') \( xP^iY_i \) and (3') for some \( l < k \) where \( Y_i \) is chosen by the \( k \)-th group of seats of the school \( s \) given the offer set \( Y_s \) such that \( f^s(l) = t \) we have \( r_l > 0 \) given the offer set \( Y_s \). First take the case where (1) \( x \notin Y_i \cap Y_s \), (2) \( xP^iY_i \) and (3) there exists \( l \in \{1, ..., \lambda_s\} \) such that \( f^s(l) = t \) and \( r_l > 0 \) given the offer set \( Y_s \). Consider the offer set \( Y_s \cup \{x\} \) for school \( s \). Let \( l^* \) be the minimum number among \( l \)'s that satisfy \( f^s(l) = t \) and \( r_l > 0 \) given the offer set \( Y_s \). When the contract \( x \) is added to the offer set \( Y_s \) the set of chosen contracts in all of the groups of seats \( l < l^* \) remain unchanged and hence the dynamic capacity of the group of seats \( l^* \) remains unchanged. Since \( r_{l^*} > 0 \) given the offer set \( Y_s \) and no contract of student \( i \) has been chosen by any group of seats \( l < l^* \), then q-responsiveness of the sub-choice functions implies that \( x \in c_s^l((Y_s \cup \{x\})^{l-1}, q_s^l(r_1, ..., r_{l-1}), f^s(l)) \). Hence, \( x \in C^s(Y \cup \{x\}, f^s, q_s) \) which is a contradiction that \( Y \) is a stable outcome. Now consider the second case where (1')
x \in Y_i \cap Y_s$, (2') $xP_i^iY_i$ and (3') for some $l < k$ where $Y_i$ is chosen by the $k^{th}$ group of seats of the school $s$ given the offer set $Y_s$ such that $f^s(l) = t$ we have $r_l > 0$ given the offer set $Y_s$. Let $l^*$ be the minimum among $l$'s that satisfy $l < k$, $f^s(l) = t$ and $r_l > 0$ given the offer set $Y_s$. Consider the offer set $Y_s \cup \{x\}$. Adding $x$ to the offer set $Y_s$ does not change the set of chosen contracts in any group of seats precedes $l^*$. Hence, the dynamic capacity of group $l^*$, $r_{l^*} > 0$ remains unchanged. By q-responsiveness of the sub-choice functions and the fact that no contract of student $i$ is chosen before the group $l^*$ we have $x \in c^s_{l^*}((Y_s \cup \{x\})^{l^*-1}, q^s_{l^*}(r_1, ..., r_{l^*-1}), f^s(l^*))$.

The “if” part Fix an outcome $Y$ that is feasible and non-wasteful up to the dynamic capacities and satisfies no-justified envy with other and own schools property. We will show that $Y$ is stable. Since feasibility of $Y$ implies the first two condition of stability we only need to show that there is no contract $x = (i, s, t) \in X \setminus Y$ such that $x \in C^s(Y \cup \{x\}, f^s, q_s)$ and $xP_i^iY$. We show this by contradiction.

Suppose that there is a contract $x = (i, s, t) \in X \setminus Y$ such that $xP_i^iY$ and $C^s(Y \cup \{x\}, f^s, q_s)$. There are two cases to consider: $s \neq s(Y_i)$ and $s = s(Y_i)$. We first consider the case $s \neq s(Y_i)$. $x \in C^s(Y \cup \{x\}, f^s, q_s)$ implies that there exists $l \in \{1, ..., \lambda_s\}$ such that $f^s(l) = t$ and $x \in c^s_l((Y_s \cup \{x\})^{l-1}, q^s_{l}(r_1, ..., r_{l-1}), t)$. Since sub-choice functions are q-responsive $x \in c^s_l((Y_s \cup \{x\})^{l-1}, q^s_{l}(r_1, ..., r_{l-1}), t)$ is possible if either $r_l > 0$ given the offer set $Y_s$ or there exists $y \in c^s_l((Y_s \cup \{x\})^{l-1}, q^s_{l}(r_1, ..., r_{l-1}), t)$ such that $i\pi^s_i(y)$. If it is the former case, then $Y$ is not non-wasteful up to the dynamic capacity. If it is the latter case, then $Y$ violates the no-justified-envy with other schools condition. In either case, this yields a contradiction. Then, it must be the case that $s = s(Y_i)$.

Let $k$ be the group of seats of school $s$ that chooses the contract $Y_i$. Note that $x \in C^s(Y \cup \{x\}, f^s, q_s)$ implies that there exists $l \in \{1, ..., \lambda_s\}$ such that $l < k$, $f^s(l) = t$ and $x \in c^s_l((Y_s \cup \{x\})^{l-1}, q^s_{l}(r_1, ..., r_{l-1}), t)$. Since the overall choice function of school $s$ respects the precedence order $f^s$ and the remaining contracts of students are removed for the rest of the choice procedure when a contract of a student is chosen by a group of slots, it must be the case that $l < k$. By q-responsiveness of sub-choice functions $x \in c^s_l((Y_s \cup \{x\})^{l-1}, q^s_{l}(r_1, ..., r_{l-1}), t)$ is possible if either $r_l > 0$ given the offer set $Y_s$ or there exists a $y \in c^s_l((Y_s \cup \{x\})^{l-1}, q^s_{l}(r_1, ..., r_{l-1}), t)$ such that $i\pi^s_i(y)$. If it is the former, then $Y$ cannot be non-wasteful up to the dynamic capacities. If it is the latter, then $Y$ does not satisfy the no-justified-envy with own schools condition. In both cases, we have a contradiction.

Proof of Proposition 3

Take a set of contracts $Y \subseteq X$ and a contract $z = (i, s, t) \in X \setminus Y$ such that $z \notin C^s(Y_s \cup \{z\}, f^s, q_s)$ where $C^s(\cdot, f^s, q_s)$ is a dynamic reserves choice function of school $s$.
given the precedence sequence $f^s$ and capacity transfer scheme $q_s$. We need to show that $z \notin C^s(Y_s, f^s, q_s)$. Consider two choice processes of school $s$ beginning with the offer sets $Y_s$ and $Y_s \cup \{z\}$, respectively. Let $Y_s^0 \equiv Y_s^0 = Y_s \cup \{z\}$. Let $Y_{s}^{k-1}$ and $(Y_s \cup \{z\})^{k-1}$ be the sets of remaining contracts for group of seats $k \in \{2, ..., \lambda_s\}$ given the initial offer sets $Y_{s}^0$ and $(Y_s \cup \{z\})^0$, respectively. Also, let $r_k$ and $\tilde{r}_k$ be the number of vacant slots in group of seats $k \in \{1, ..., \lambda_s - 1\}$ given the initial offer sets $Y_{s}^0$ and $(Y_s \cup \{z\})^0$, respectively.

For the first group of slots $z \notin c_1^s((Y_s \cup \{z\})^0, \tilde{q}_s^1, t)$ implies that $z \notin c_1^s(Y_s^0, \tilde{q}_s^1, t)$ since the sub-choice function $c_1^s(\cdot, \tilde{q}_s^1, f^s(1))$ is q-responsive, and hence satisfies the IRC condition. Moreover, $c_1^s((Y_s \cup \{z\})^0, \tilde{q}_s^1, f^s(1)) = c_1^s(Y_s^0, \tilde{q}_s^1, f^s(1))$ by the IRC of sub-choice functions. Furthermore, $(Y_s \cup \{z\})^1 = Y_s^1 \cup \{z\}$ if no other contract of student $i$ is chosen and $(Y_s \cup \{z\})^1 = Y_s^1$ if some other contract of student $i$ is chosen because contracts of the same sets of students are removed from $Y_s^0$ and $(Y_s \cup \{z\})^0$, respectively. Also, $r_1 = \tilde{r}_1$ since the chosen set has the same size under both offer processes.

Now suppose that for all $l \in \{2, ..., k\}$, the followings hold: (1) $(Y \cup \{z\})^l = Y^l$ if some other contract of student $i$ than $z$ at or before group of seats $l-1$ is chosen, (2) $(Y \cup \{z\})^l = Y^l \cup \{z\}$ if some other contract of student $i$ than $z$ at or before in group of seats $l-1$ and, (3) $r_{l-1} = \tilde{r}_{l-1}$.

Consider sub-choice function of group of seats $k$. Since $r_{l-1} = \tilde{r}_{l-1}$ holds for all $l = 2, ..., k$ we will have $q^k_s = q^k_s(r_1, ..., r_{k-1}) = q^k_s(\tilde{r}_1, ..., \tilde{r}_{k-1})$. Hence, group of seats $k$ has the same dynamic capacity under offer processes $Y_s$ and $Y_s \cup \{z\}$.

If some other contract of student $i$ than $z$ is chosen at or before group of seats $k$, then we will have $c_k^s((Y_s \cup \{z\})^{k-1}, q^k_s(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) = c_k^s(Y_s^k, q^k_s(r_1, ..., r_{k-1}), f^s(k))$ since the sub-choice function is q-responsive and by the inductive assumption (1). The same set of contracts will be chosen under both offer processes. Also, the same set of contracts will be removed from the offer sets $Y_s^k$ and $(Y_s \cup \{z\})^k$, which implies that $Y_s^{k+1} = (Y_s \cup \{z\})^{k+1}$ in this case. Also, $r_k = \tilde{r}_k$ since group $k$ has the same dynamic capacities under both offer processes and the size of the chosen set for group $k$ is the same.

If some other contract of $i$ than $z$ is not chosen at or before group of seats $k$, then the fact that $z \notin C^s(Y_s \cup \{z\}, f^s, q_s)$ together with our inductive assumption (2) imply that $c_k^s((Y_s \cup \{z\})^{k-1}, q^k_s(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) = c_k^s(Y_s^k, q^k_s(r_1, ..., r_{k-1}), f^s(k))$ given that the sub-choice function $c_k^s(\cdot, q^k_s, f^s(k))$ is q-responsive. Then, the same of contracts will be removed both from $Y_s^k$ and $(Y_s \cup \{z\})^k$. Hence, $(Y_s \cup \{z\})^{k+1} = Y_s^{k+1} \cup \{z\}$. Also, since the dynamic capacity of the group of seats $k$ and the size of the chosen set under both offer processes are the same, we will have $r_k = \tilde{r}_k$.

We show that under offer processes $Y_s$ and $Y_s \cup \{z\}$ the same set of contracts are chosen in every group of seats. Therefore, $z \notin C^s(Y_s \cup \{z\}, f^s, q_s)$ implies $z \notin C^s(Y_s \cup \{z\}, f^s, q_s)$.
Proof of Proposition 5

Consider a set of contracts $Y \subseteq X$ and contracts $x, z \in X \setminus Y$ such that $i(x), i(z) \notin i(Y)$. Suppose also that $z \notin C^*(Y_s \cup \{z\}, f^s, q_s)$. We need to show that $z \notin C^*(Y_s \cup \{x, z\}, f^s, q_s)$. There are two cases to consider:

**Case 1** $x \notin C^*(Y_s \cup \{x, z\}, f^s, q_s)$.

By Proposition 3, overall choice functions satisfy the IRC condition. Hence, $x \notin C^*(Y_s \cup \{x, z\}, f^s, q_s)$ implies $C^*(Y_s \cup \{x, z\}, f^s, q_s) = C^*(Y_s \cup \{z\}, f^s, q_s)$. Thus, $z \notin C^*(Y_s \cup \{x, z\}, f^s, q_s)$ by our assumption.

**Case 2** $x \in C^*(Y_s \cup \{x, z\}, f^s, q_s)$.

Set $Z^0 \equiv Y_s \cup \{x\}$ and $\tilde{Z}^0 \equiv Y_s \cup \{x, z\}$. Let $Z_k \equiv c_k^s(Z^{k-1}, q^s_k(r_1, \ldots, r_{k-1}), f^s(k))$ and $\tilde{Z}_k \equiv c_k^s(\tilde{Z}^{k-1}, q^s_k(\tilde{r}_1, \ldots, \tilde{r}_{k-1}), f^s(k))$ where $Z^{k-1}$ and $\tilde{Z}^{k-1}$ are the set of remaining contracts after group of seats $(k - 1)$ selects under offer processes beginning with $Z^0$ and $\tilde{Z}^0$, respectively and $r_i$ and $\tilde{r}_i$ are the number of vacant slots in group of seats $l \in \{1, \ldots, k - 1\}$ given the initial offer sets $Z^0$ and $\tilde{Z}^0$, respectively.

Since $x \in C^*(Y_s \cup \{x, z\}, f^s, q_s)$ there exists $j \in \{1, \ldots, \lambda_s\}$ be such that $x \in c_j^s(\tilde{Z}^{j-1}, q^s_j(\tilde{r}_1, \ldots, \tilde{r}_{j-1}), f^s(j))$. For all $l \in \{1, \ldots, j-1\}$, $x \notin c_l^s(\tilde{Z}^{l-1}, q^s_l(\tilde{r}_1, \ldots, \tilde{r}_{l-1}), f^s(l))$. We also know that $z \notin c_l^s(Z^{l-1}, q^s_l(r_1, \ldots, r_{l-1}), f^s(l))$ for all $l = 1, \ldots, \lambda_s$ since $z \notin C^*(Y_s \cup \{x\})$. Then, since sub-choice functions are q-responsive, $z \notin c_l^s(\tilde{Z}^{l-1}, q^s_l(\tilde{r}_1, \ldots, \tilde{r}_{l-1}), f^s(l))$.

Furthermore, we have $\tilde{Z}^l = Z^l \cup \{x\}$ and $z \in Z^l$ for all $l = 1, \ldots, j-1$. Since $r_i = \tilde{r}_i$ for all $l = 1, \ldots, j-1$, we have $q^s_l(r_1, \ldots, r_{j-1}) = q^s_l(\tilde{r}_1, \ldots, \tilde{r}_{j-1})$. By the q-responsive of sub-choice functions and the fact that $x \in c_j^s(\tilde{Z}^{j-1}, q^s_j(\tilde{r}_1, \ldots, \tilde{r}_{j-1}), f^s(j))$, we have $Z^j \subseteq \tilde{Z}^j$. Given that $|Z^{j-1}| = |\tilde{Z}^{j-1}|$ and sub-choice function is q-responsive (which implies the LAD) we have $|c_j^s(\tilde{Z}^{j-1}, q^s_j(\tilde{r}_1, \ldots, \tilde{r}_{j-1}), f^s(j))| \geq |c_j^s(Z^{j-1}, q^s_j(r_1, \ldots, r_{j-1}), f^s(j))|$. This implies $q^s_{j+1}(r_1, \ldots, r_j) \geq q^s_{j+1}(\tilde{r}_1, \ldots, \tilde{r}_j)$ by the monotonicity of capacity transfer schemes. By our initial assumption, we know that $z \notin c_{j+1}^s(Z^j, q^s_{j+1}(r_1, \ldots, r_j), f^s(j+1))$.

This implies that $z \notin c_{j+1}^s(\tilde{Z}^j, q^s_{j+1}(\tilde{r}_1, \ldots, \tilde{r}_j), f^s(j+1))$ because the sub-choice function is q-responsive, $Z^j \subseteq \tilde{Z}^j$ and $q^s_{j+1}(r_1, \ldots, r_j) \geq q^s_{j+1}(\tilde{r}_1, \ldots, \tilde{r}_j)$. Moreover, $Z^{j+1} \subseteq \tilde{Z}^{j+1}$.

To see why, consider $y \in Z^{j+1}$. It means $y \notin c_{j+1}^s(Z^j, q^s_{j+1}(r_1, \ldots, r_j), f^s(j+1))$. Sub-choice function $c_j^s(\cdot, \cdot, \cdot)$ being q-responsive implies $y \notin c_{j+1}^s(\tilde{Z}^j, q^s_{j+1}(\tilde{r}_1, \ldots, \tilde{r}_j), f^s(j+1))$. Hence, $y \in \tilde{Z}^{j+1}$. Also, $Z^j \subseteq \tilde{Z}^j$, $q^s_{j+1}(r_1, \ldots, r_j) \geq q^s_{j+1}(\tilde{r}_1, \ldots, \tilde{r}_j)$ and q-responsiveness of sub-choice functions imply that $\tilde{r}_{j+1} \leq r_{j+1}$. Intuitively, it means that the number of vacant seats becomes lower when the offer set expands and the capacity is decreased.

Now, suppose that for all $l = j, \ldots, k - 1$ we have $z \notin c_l^s(\tilde{Z}^{l-1}, q^s_l(\tilde{r}_1, \ldots, \tilde{r}_{l-1}), f^s(l))$, $Z^l \subseteq \tilde{Z}^l$ and $\tilde{r}_{l+1} \leq r_{l+1}$ (We have already shown it for $l = j$). Now we will show
them for \( l = k \). By our inductive assumptions we have the followings: \( Z^{k-1} \subseteq \hat{Z}^{k-1}, q^k_s(r_1, ..., r_{k-1}) \geq q^k_s(\tilde{r}_1, ..., \tilde{r}_{k-1}) \). Also, our initial assumption \( z \notin C^s(Y_s \cup \{z\}, f^s, q_s) \) implies \( z \notin c^k_s(\bar{Z}^{k-1}, q^k_s(r_1, ..., r_{k-1}), f^s(l)) \). Since the sub-choice function of group of seats \( k \) is \( q \)-responsive this, in turn, implies that \( z \notin c^k_s(\hat{Z}^{k-1}, q^k_s(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) \). To see \( Z^k \subseteq \hat{Z}^k \) consider a \( y \in Z^k \). It means \( y \notin c^k_s(\hat{Z}^{k-1}, q^k_s(r_1, ..., r_{k-1}), f^s(k)) \). Sub-choice function being \( q \)-responsive implies that if \( y \) is not chosen from a smaller offer set when the capacity is higher, then it will not be chosen from a larger set when the capacity is reduced. That is, \( y \notin c^k_s(\hat{Z}^{k-1}, q^k_s(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) \). Therefore, \( y \in \hat{Z}^k \). Finally, we obtain \( \tilde{r}_{k+1} \leq r_{k+1} \) because \( Z^k \subseteq \hat{Z}^k \) and \( q^k_s(r_1, ..., r_k) \geq q^k_s(r_1, ..., r_k) \) together with the fact that the sub-choice function of group of seats \( (k + 1) \) is \( q \)-responsive.

We can now conclude that \( z \notin C^s(Y_s \cup \{z\}, f^s, q_s) \) implies \( z \notin C^s(Y_s \cup \{x, z\}, f^s, q_s) \).

**Proof of Proposition 7**

Let \( f^s \) and \( q_s \) be the precedence sequence and capacity transfer scheme of school \( s \in S \), respectively. Take an offer set \( Y = Y^0 \subseteq X \) and assume there is no pair of contracts \( z, z' \in Y^0 \) such that \( i(z) = i(z') \) and \( z, z' \in \overline{C}^s(Y, f^s, q_s) \). We want to show that \( \overline{C}^s(Y, f^s, q_s) = C^s(Y, f^s, q_s) \).

Let \( Y_j \) and \( Y^j \) be the set of contracts that are chosen by group of seats \( j \) and set of contracts that remains in the choice procedure after group of seats \( j \) selects according to dynamic reserve choice function \( C(\cdot) \), respectively. Similarly, let \( \overline{Y}_j \) and \( \overline{Y}^j \) be the set of contracts that are chosen by group of seats \( j \) and set of contracts that remains in the choice procedure after group of seats \( j \) selects according to the completion of the dynamic reserve choice function \( \overline{C}(\cdot) \), respectively. Notice that \( Y^0 = \overline{Y}^0 \). Let \( r_j \) and \( \overline{r}_j \) be the number of vacant seats in group of seats \( j \) in the choice procedures \( C^s(Y, f^s, q_s) \) and \( \overline{C}^s(Y, f^s, q_s) \), respectively. Also, let \( q^j_s(r_1, ..., r_{j-1}) \) and \( \overline{q}^j_s(\overline{r}_1, ..., \overline{r}_{j-1}) \) denote the dynamic capacities of group of seats \( j \) under choice procedures \( C^s(Y, f^s, q_s) \) and \( \overline{C}^s(Y, f^s, q_s) \), respectively.

Given \( \overline{q}^1_s \) and \( Y^0 = \overline{Y}^0 \), we have \( \overline{Y}_1 = c^1_s(Y^0, \overline{q}^1_s, f^s(1)) = Y_1 \) by the construction of \( \overline{C}^s \). Moreover, \( \overline{r}_1 = r_1 \) and \( \overline{q}^2_s(\overline{r}_1) = q^2_s(r_1) \).

Suppose that for all \( j \in \{2, ..., k-1\} \) we have that \( Y_j = \overline{Y}_j \). We need to show that it holds for group of seats \( k \), i.e., \( Y_k = \overline{Y}_k \). Since the chosen set is the same in every group seats from 1 to \( k-1 \) under the dynamic reserves choice function and its completion, the number of remaining slots in each group is the same as well. Then, the dynamic capacity of the group of seats \( k \) are the same under choice procedures \( C^s(Y, f^s, q_s) \) and \( \overline{C}^s(Y, f^s, q_s) \), i.e., \( q^k_s(r_1, ..., r_{k-1}) = \overline{q}^k_s(\overline{r}_1, ..., \overline{r}_{k-1}) \). Since there are no two contracts of an agent chosen by \( \overline{C}^s(Y, f^s, q_s) \), one can deduce that all of the remaining contracts of agents whose contracts are chosen by previous sub-choice functions are rejected by
that \( c^s_k(Y^{k-1}, q^k_s(r_1, \ldots, r_{k-1}), f^s(k)) \). Therefore, the IRC of sub-choice function imply that

\[
c^s_k(Y^{k-1}, q^k_s(r_1, \ldots, r_{k-1}), f^s(k)) = c^s_k(Y^{k-1}, q^k_s(r_1, \ldots, r_{k-1}), f^s(k)).
\]

Hence, we have \( Y_k = Y_k, r_k = r_k, \) and \( q^{k+1}_s(r_1, \ldots, r_k) = q^{k+1}_s(r_1, \ldots, r_k) \).

Thus, since in each group of seats the same sets of contracts are chosen by the dynamic reserve choice function and its completion, the result follows.

**Proof of Proposition 8**

For any \( Y \subseteq X \) such that \( Y \neq \overline{C}^s(Y, f^s, q_s) \), let \( x \) be one of the rejected contracts, i.e., \( x \in Y \setminus \overline{C}^s(Y, f^s, q_s) \). To show that the IRC is satisfied, we need to prove that \( \overline{C}^s(Y, f^s, q_s) = \overline{C}(Y \setminus \{x\}, f^s, q_s) \). Let \( \tilde{Y} = Y \setminus \{x\} \). Suppose \( (Y_j, \tilde{r}_j, \tilde{f}_j) \) and \( (\tilde{Y}_j, \tilde{r}_j, \tilde{f}_j) \) denote the sequence of the set of chosen contracts, the number of vacant slots, and the remaining set of contracts for group of seats \( j = 1, \ldots, \lambda_s \) from \( Y \) and \( \tilde{Y} \) under the completion of a dynamic reserve choice function, respectively.

For the first group of seats, since the sub-choice functions satisfy the IRC, we have \( \overline{Y}_1 = \tilde{Y}_1 \). Moreover, \( \tilde{r}_1 = r_1 \) and \( \overline{Y}^1_1 \setminus \{x\} = \tilde{Y}^1_1 \). By induction, for each \( j = 2, \ldots, k-1 \), assume that \( \overline{Y}_j = \tilde{Y}_j, \tilde{r}_j = \tilde{r}_j, \) and \( \overline{Y}^j_1 \setminus \{x\} = \tilde{Y}^j_1 \).

We need to show them for the group seats \( k \). Since \( x \notin \overline{C}^s(Y, f^s, q_s) \) and the sub-choice functions satisfy the IRC condition we have

\[
c^s_k(\overline{Y}^{k-1}, q^k_s(r_1, \ldots, r_{k-1}), f^s(k)) = c^s_k(\tilde{Y}^{k-1}, q^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1}), f^s(k)).
\]

The same set of contracts are chosen for group of seats \( k \) in the choice processes beginning with \( Y \) and \( Y \cup \{x\} \), respectively. By our inductive assumption that \( \tilde{r}_j = \tilde{r}_j \) for each \( j = 2, \ldots, k-1 \), the dynamic capacity of group of seats \( k \) are the same under choice processes beginning with \( Y \) and \( Y \cup \{x\} \), respectively. Then, since the chosen sets are the same in the choice processes beginning with \( Y \) and \( Y \cup \{x\} \), the number of remaining seats will be the same, as well, i.e., \( \tilde{r}_k = \tilde{r}_k \). Finally, since the chosen sets are the same and we know that \( x \) will not be chosen from the set \( \tilde{Y}^{k-1} \cup \{x\} \), then we have \( \tilde{Y}^k = \tilde{Y}^k \cup \{x\} \).

Since for all \( j \in \{1, \ldots, \lambda_s\} \), \( Y_j = \tilde{Y}_j \), we have \( \overline{C}(Y, f^s, q_s) = \overline{C}(\tilde{Y}, f^s, q_s) \). Hence, \( \overline{C}^s(\cdot, f^s, q_s) \) satisfies the IRC.

**Proof of Proposition 9**

Consider an offer set \( Y \subseteq X \) such that \( Y \neq \overline{C}^s(Y, f^s, q_s) \). Let \( x \) be one of the rejected contracts, i.e., \( x \in Y \setminus \overline{C}(f^s, q_s) \) and let \( z \) be an arbitrary contract in \( X \setminus Y \). To show substitutability, we need to show that \( x \notin \overline{C}(Y \cup \{z\}, f^s, q_s) \). Let \( \tilde{Y} = Y \cup \{z\} \).
Suppose \((Y_j, r_j, Y^j)\) and \((\tilde{Y}_j, \tilde{r}_j, \tilde{Y}^j)\) denote the sequence of the set chosen contracts, the number of vacant slots, and the set of remaining contracts for group of seats \(j = 1, \ldots, \lambda_s\) from \(Y\) and \(\tilde{Y}\), respectively, under the completion of a dynamic reserve choice function. There are two cases to consider:

Case 1 \(z \in \tilde{Y} \setminus C^s(\tilde{Y}, f^s, q^s)\).

In this case, the IRC of \(C^s\) implies \(C^s(\tilde{Y}, f^s, q^s) = C^s(Y, f^s, q^s)\). Therefore, \(x \notin C^s(\tilde{Y}, f^s, q^s)\).

Case 2 \(z \in C^s(\tilde{Y}, f^s, q^s)\).

Let \(j\) be the group of seats such that \(z \in \tilde{Y}_j\). By the IRC of sub-choice functions, \(x \notin \tilde{Y}_j = Y_j\), for all \(j' = 1, \ldots, j - 1\). Moreover, \(\tilde{Y}^{j-1} = Y^{j-1} \cup \{z\}\) and \(\tilde{r}_{j'} = r_{j'}\), for all \(j' = 1, \ldots, j - 1\).

For the group of seats \(j\), first notice that the dynamic capacity is the same under choice procedures beginning with \(Y = Y^0\) and \(Y \cup \{z\} = \tilde{Y}^0\), respectively, because the number of unused slots from group of seats 1 to \(j - 1\) are the same under two choice procedures. We know that \(z\) is chosen exactly at group of seats \(\tilde{Y}_j\) in the offer process beginning with \(\tilde{Y}^0\). There are two cases here:

(a) The dynamic capacity of group of seats \(j\) is exhausted in the choice process beginning with \(Y^0\). In this case, by choosing \(z\) from \(\tilde{Y}^0\) another contract, call it \(y \in \tilde{Y}^0\) is rejected even though \(y\) was chosen at group of seats \(j\) in the choice procedure beginning with \(Y^0\).

(b) The dynamic capacity of group of seats \(j\) is not exhausted in the choice process beginning with \(Y^0\). In this case, \(z\) is chosen at group of seats \(j\) in the offer process beginning with \(\tilde{Y}^0\) without rejecting any contract that was chosen in the choice process beginning with \(Y^0\) at the group of seats \(j\).

In the case of (a),

\[ |c^s_j(Y^{j-1}, q^s_j(r_1, \ldots, r_{j-1}), f^s(j))| = q^s_j(r_1, \ldots, r_{j-1}) \]

and \(z \in c^s_j(\tilde{Y}^{j-1}, q^s_j(r_1, \ldots, r_{j-1}), f^s(j))\) implies that there exists

\[ y \in c^s_j(Y^{j-1}, q^s_j(r_1, \ldots, r_{j-1}), f^s(j)) \setminus c^s_j(\tilde{Y}^{j-1}, q^s_j(\tilde{r}_1, \ldots, \tilde{r}_{j-1}), f^s(j)) \]

This, in turn, implies that \(\tilde{Y}^j = Y^j \cup \{y\}\). Since the capacity of group of seats \(j\) is exhausted under both choice processes, the number of vacant seats for group \(j\) will be 0 under choice both choice processes. Hence, the capacity of the group of seats \(j + 1\) will be the same under both choice processes.

Notice that

\[ x \notin Y_j \implies x \notin \tilde{Y}_j \]
because
\[ c_s^j(Y^{j-1}, q^j_s(r_1, ..., r_{j-1}), f^s(j)) \cup \{ z \} \setminus \{ y \} = c_s^j(\tilde{Y}^{j-1}, q^j_s(\tilde{r}_1, ..., \tilde{r}_{j-1}), f^s(j)) \]

In case (b), we have \( |c_s^j(Y^{j-1}, q^j_s(r_1, ..., r_{j-1}), f^s(j))| < q^j_s(r_1, ..., r_{j-1}) \). Hence \( r_j > 0 \).

Then, since sub-choice functions are responsive, we will have
\[ c_s^j(\tilde{Y}^{j-1}, q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}), f^s(j)) = \{ z \} \cup c_s^j(Y^{j-1}, q^j_s(r_1, ..., r_{j-1}), f^s(j)) \]

Hence,
\[ x \in Y_j \implies x \notin \tilde{Y}_j \]

We also have \( r_j = \tilde{r}_j + 1 \). Moreover, the set of remaining contracts under both choice processes will be the same, i.e., \( \tilde{Y}^j = Y^j \). The facts \( r_j' = \tilde{r}_j \) for all \( j' = 1, ..., j - 1 \) and \( r_j = \tilde{r}_j + 1 \) implies, by the monotonicity of capacity transfer schemes, that either \( q_s^{j+1}(r_1, ..., r_j) = q_s^{j+1}(\tilde{r}_1, ..., \tilde{r}_j) \) or \( q_s^{j+1}(r_1, ..., r_j) = 1 + q_s^{j+1}(\tilde{r}_1, ..., \tilde{r}_j) \) hold.

Suppose now that for all \( \gamma = j, ..., k - 1 \) we have that either
\[ \tilde{Y}^\gamma = Y^\gamma \cup \{ \tilde{y} \} \text{ for some } \tilde{y} \text{ and } q_s^{\gamma+1}(\tilde{r}_1, ..., \tilde{r}_\gamma) = q_s^{\gamma+1}(r_1, ..., r_\gamma) \]
or
\[ \tilde{Y}^\gamma = Y^\gamma \text{ and } q_s^{\gamma+1}(\tilde{r}_1, ..., \tilde{r}_\gamma) \leq q_s^{\gamma+1}(r_1, ..., r_\gamma) \leq 1 + q_s^{\gamma+1}(\tilde{r}_1, ..., \tilde{r}_\gamma \} \].

We have already shown it for \( \gamma = j \). We will now show that it also hold for \( \gamma = k \).

We will first analyze the former case. By inductive assumption we have \( \tilde{Y}^{k-1} = Y^{k-1} \cup \{ \tilde{y} \} \) for some contract \( \tilde{y} \). If \( \tilde{y} \) is not chosen from the set \( \tilde{Y}^{k-1} \) then exactly same set of contracts will be chosen from \( Y^{k-1} \) and \( \tilde{Y}^{k-1} \) since the capacities of group \( k \) are the same under both choice processes and the sub-choice function satisfies the IRC condition. Then, we will have \( \tilde{Y}^k = Y^k \cup \{ \tilde{y} \} \). Moreover, since the number of vacant seats at group \( k \) will be the same under both processes, we will have \( q_s^{k+1}(r_1, ..., r_k) = q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \). If \( \tilde{y} \) is chosen from the set \( \tilde{Y}^{k-1} \) we have two sub-cases depending on the dynamic capacity of group \( k \) is exhausted under the choice process beginning with \( Y^0 \). If it is not exhausted, then we will have
\[ c_s^k(\tilde{Y}^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k)) = \{ \tilde{y} \} \cup c_s^k(Y^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(k)) \]
which implies that \( \tilde{Y}^k = Y^k \). Moreover, we will have \( r_k = \tilde{r}_k + 1 \). Monotonicity of capacity transfer scheme implies that
\[ q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \leq q_s^{k+1}(r_1, ..., r_k) \leq 1 + q_s^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \].

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The first inequality follows from the fact that $\tilde{r}_i \leq r_i$ for all $i = 1, \ldots, k$. The second inequality follows from the second condition of the monotonicity of the capacity transfer schemes which was

$$\sum_{m=1}^{k-1} \left[ q^m_s(\tilde{r}_1, \ldots, \tilde{r}_{m-1}) - q^m_s(r_1, \ldots, r_{m-1}) \right] \leq \sum_{m=1}^{k-1} [\tilde{r}_m - r_m].$$

On the other hand, if the dynamic capacity of group $k$ is exhausted in the choice procedure beginning with $Y^0$, i.e., $|c_k^s(Y^{k-1}, q^k_s(r_1, \ldots, r_{k-1}), f^s(j))| = q^k_s(r_1, \ldots, r_{k-1})$ then choosing $\tilde{y}$ from the set $\tilde{Y}^{k-1}$ implies that there exists a contract $\tilde{y}$ that is chosen from $Y^{k-1}$ but rejected from $\tilde{Y}^{k-1}$. Then, we will have $\tilde{Y}^k = Y^k \cup \{\tilde{y}\}$ since the sub-choice function is $q$-responsive and capacities of group $k$ are the same under both choice processes. In this case, we will have $r_k = \tilde{r}_k = 0$. Since $\tilde{r}_i \leq r_i$ for all $i = 1, \ldots, k$ we will have $q^{k+1}_s(\tilde{r}_1, \ldots, \tilde{r}_k) \leq q^{k+1}_s(r_1, \ldots, r_k)$ from the first condition of the monotonicity of the capacity transfer scheme. Since $q^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1}) = q^k_s(r_1, \ldots, r_{k-1})$ and $\tilde{r}_k = r_k$ we will have $q^{k+1}_s(\tilde{r}_1, \ldots, \tilde{r}_k) \geq q^{k+1}_s(r_1, \ldots, r_k)$ by the second condition of the monotonicity of capacity transfer schemes.

We will now analyze the latter case where we have $\tilde{Y}^{k-1} = Y^{k-1}$ and either $q^k_s(r_1, \ldots, r_{k-1}) = \tilde{q}^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1})$ or $\tilde{q}^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1}) = 1 + q^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1})$. If $q^k_s(r_1, \ldots, r_{k-1}) = q^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1})$ then given that $\tilde{Y}^{k-1} = Y^{k-1}$ we have $\tilde{Y}^k = Y^k$. This also implies $r_k = \tilde{r}_k$. Moreover, by the monotonicity of capacity transfer scheme we obtain $q^{k+1}_s(r_1, \ldots, r_k) = q^{k+1}_s(\tilde{r}_1, \ldots, \tilde{r}_k)$. Because $\tilde{r}_i \leq r_i$ for all $i = 1, \ldots, k$ implies $q^{k+1}_s(r_1, \ldots, r_k) \geq q^{k+1}_s(\tilde{r}_1, \ldots, \tilde{r}_k)$ by the first condition of monotonicity of capacity transfers. The second condition of monotonicity of capacity transfers implies $q^{k+1}_s(r_1, \ldots, r_k) \leq q^{k+1}_s(\tilde{r}_1, \ldots, \tilde{r}_k)$. If $q^k_s(r_1, \ldots, r_{k-1}) = 1 + q^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1})$ then, given $\tilde{Y}^{k-1} = Y^{k-1}$, we have two sub-cases here. If

$$c^s_k(\tilde{Y}^{k-1}, q^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1}), f^s(k)) = c^s_k(Y^{k-1}, q^k_s(r_1, \ldots, r_{k-1}), f^s(k))$$

then we will have $\tilde{Y}^k = Y^k$. Also, monotonicity of capacity transfer scheme implies that

$$q^{k+1}_s(\tilde{r}_1, \ldots, \tilde{r}_k) \leq q^{k+1}_s(r_1, \ldots, r_k) \leq 1 + q^{k+1}_s(\tilde{r}_1, \ldots, \tilde{r}_k).$$

If

$$c^s_k(\tilde{Y}^{k-1}, q^k_s(\tilde{r}_1, \ldots, \tilde{r}_{k-1}), f^s(k)) \cup \{y^*\} = c^s_k(Y^{k-1}, q^k_s(r_1, \ldots, r_{k-1}), f^s(k))$$

\footnote{In the second condition of the monotonicity of the capacity transfer schemes, if the number of vacant seats is written as the dynamic capacity of the group minus the number of chosen contracts then we will have the following: the dynamic capacity of the group $k+1$ in the choice process beginning with $Y$ minus the dynamic capacity of the group $k+1$ in the choice process beginning with $Y \cup \{z\} = \tilde{Y}^0$ must be less than or equal to summation of the difference of the number of chosen contracts from group 1 to group $k$, which is 0 in this specific case.}
for some \( y^* \) then we will have \( \bar{Y}^k = Y^k \cup \{ y^* \} \). Moreover, monotonicity of capacity transfer schemes in this case implies that

\[
q_s^{k+1}(r_1, \ldots, r_k) = q_s^{k+1}(\bar{r}_1, \ldots, \bar{r}_k)
\]

because given \( \bar{r}_i \leq r_i \) for all \( i = 1, \ldots, k \) the first condition of monotonicity of the capacity transfers implies that \( q_s^k(r_1, \ldots, r_k) \geq q_s^k(\bar{r}_1, \ldots, \bar{r}_k) \). On the other hand, the second condition of the monotonicity of the capacity transfers imply that \( q_s^k(r_1, \ldots, r_{k-1}) \) and \( q_s^k(\bar{r}_1, \ldots, \bar{r}_{k-1}) \),

Since \( x \notin Y_k \) we will have \( x \notin \bar{Y}_k \) for all \( k = 1, \ldots, \lambda_s \). Thus, we can conclude that \( x \notin \bar{C}^s(Y \cup \{ z \}, f^s, q_s) \) which tells that the completion \( \bar{C}^s \) satisfies substitutability condition.

**Proof of Proposition 10**

Consider two sets of contracts \( Y \) and \( \bar{Y} \) such that \( Y \subseteq \bar{Y} \subseteq X \). Let \( f^s \) and \( q_s \) be the precedence sequence and capacity transfer scheme of school \( s \in S \). We want to show that \( | \bar{C}^s(Y, f_s, q_s) | \leq | \bar{C}^s(\bar{Y}, f^s, q_s) | \)

Let \( (Y_j, r_j, Y^j) \) and \( (\bar{Y}_j, \bar{r}_j, \bar{Y}^j) \) be the sequences of sets of chosen contracts, numbers of vacant seats and sets of remaining contracts for group of seats \( j = 1, \ldots, \lambda_s \) under choice processes beginning with \( Y = Y^0 \) and \( \bar{Y}^0 = \bar{Y} \), respectively.

For the first group of seats with capacity \( \bar{q}_s^1 \); since the sub-choice function is \( q \)-responsive (which implies the LAD) we have

\[
| Y_1 | = | c_1^s(Y^0, \bar{q}_s^1, f^s(1)) | \leq | c_1^s(\bar{Y}^0, \bar{q}_s^1, f^s(1)) | = | \bar{Y}_1 | .
\]

Then, it implies that \( r_1 = \bar{q}_s^1 - | Y_1 | \geq \bar{r}_1 = \bar{q}_s^1 - | \bar{Y}_1 | \). Moreover, we have \( Y^1 \subseteq \bar{Y}^1 \). To see it consider a \( y \in Y^1 \). It means \( y \notin Y_1 \). If \( y \) is not chosen from a smaller set \( Y^0 \) then it cannot be chosen from a larger set \( \bar{Y}^0 \) because sub-choice function is \( q \)-responsive (hence, substitutable).

Suppose that \( \bar{r}_j \leq r_j \) and \( Y^j \subseteq \bar{Y}^j \) hold for all \( j = 1, \ldots, k-1 \). We need to show that both of them hold for group of seat \( k \).

Given that \( \bar{r}_j \leq r_j \) for all \( j = 1, \ldots, k-1 \) the first condition of monotonicity of capacity transfer schemes imply that \( q_s^k(r_1, \ldots, r_{k-1}) \geq q_s^k(\bar{r}_1, \ldots, \bar{r}_{k-1}) \). The second condition of monotonicity of capacity transfer scheme puts an upper bound for the difference between \( q_s^k(r_1, \ldots, r_{k-1}) \) and \( q_s^k(\bar{r}_1, \ldots, \bar{r}_{k-1}) \), i.e., \( \sum_{j=1}^k [q_s^j(r_1, \ldots, r_{j-1}) - q_s^j(\bar{r}_1, \ldots, \bar{r}_{j-1})] \leq \sum_{j=1}^{k-1} [r_j - \bar{r}_j] \). For the group of seats \( k \)

\[
| Y_k | - | \bar{Y}_k | \leq | Y_k | - | c_k^s(Y^{k-1}, \bar{q}_s^k(\bar{r}_1, \ldots, \bar{r}_{k-1}, f^s(k)) | .
\]
because $|\tilde{Y}_k| = |c^s_k(\tilde{Y}^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k))| \geq |c^s_k(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k))|$ by the q-responsiveness of the sub-choice function. We, then, have

$$|Y_k| - |c^s_k(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k))| \leq q_s^k(r_1, ..., r_{k-1}) - q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$$

follows from q-responsiveness because $|Y_k| - |c^s_k(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k))| \leq q_s^k(r_1, ..., r_{k-1}) - q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$ is the difference between the number of chosen contracts when the capacity is (weakly) increased from $q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$ to $q_s^k(r_1, ..., r_{k-1})$. Hence, the difference $|Y_k| - |c^s_k(Y^{k-1}, q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}), f^s(k))|$ can not exceed the increase in the capacity which is $q_s^k(r_1, ..., r_{k-1}) - q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$. Therefore, now we have

$$|Y_k| - |\tilde{Y}_k| \leq q_s^k(r_1, ..., r_{k-1}) - q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$$

rearranging gives us

$$q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}) - |\tilde{Y}_k| \leq q_s^k(r_1, ..., r_{k-1}) - |Y_k|$$

which is $\tilde{r}_k \leq r_k$.

Given that $Y^{k-1} \subseteq \tilde{Y}^{k-1}$ and $q_s^k(r_1, ..., r_{k-1}) \geq q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$ we will have $Y^k \subseteq \tilde{Y}^k$. To see why consider a contract $x \in Y^k$. That means $x \in Y^{k-1}$ but $x$ is not chosen from $Y^{k-1}$ when the capacity is $q_s^k(r_1, ..., r_{k-1})$, i.e., $x \notin c^s_k(Y^{k-1}, q_s^k(r_1, ..., r_{k-1}), f^s(x))$. Then when the capacity is reduced to $q_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1})$ and the set $Y^{k-1}$ is expanded to $\tilde{Y}^{k-1}$ $x$ cannot be chosen because the sub-choice function is q-responsive. Hence, it must be the case that $x \in \tilde{Y}^k$.

Now, let $\eta_j = r_j - \tilde{r}_j$. We know that $\eta_j \geq 0$ for all $j = 1, ..., \lambda_s$ as we just proved above. Plugging $r_j = q_s^j(r_1, ..., r_{j-1}) - |Y_j|$ and $\tilde{r}_j = q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) - |\tilde{Y}_j|$ in $\eta_j = r_j - \tilde{r}_j$ gives us

$$|\tilde{Y}_j| = q_s^j(r_1, ..., r_{j-1}) - q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) + |Y^j| + \eta_j$$

Summing both the right and left hand sides for $j = 1, ..., \lambda_s$ yields

$$\sum_{j=1}^{\lambda_s} |\tilde{Y}_j| = \sum_{j=1}^{\lambda_s} |Y_j| + \sum_{j=2}^{\lambda_s} [q_s^j(r_1, ..., r_{j-1}) - q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1})] + \sum_{j=1}^{\lambda_s} \eta_j$$

Since each $\eta_j \geq 0$ we have

$$\sum_{j=1}^{\lambda_s} |\tilde{Y}_j| \geq \sum_{j=1}^{\lambda_s} |Y_j| + \sum_{j=2}^{\lambda_s} [q_s^j(r_1, ..., r_{j-1}) - q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1})]$$
Also, we know that \( q_s^j(r_1, \ldots, r_{j-1}) \geq q_s^j(\tilde{r}_1, \ldots, \tilde{r}_{j-1}) \) for all \( j = 2, \ldots, \lambda_s \) by the first condition of the monotonicity of the capacity transfer scheme as \( r_i \geq \tilde{r}_i \) for all \( i = 1, \ldots, j - 1 \) (Notice that for \( j = 1 \) the capacity is fixed to \( q_1^s \) under both processes). Therefore, we have

\[
\sum_{j=1}^{\lambda_s} |\tilde{Y}_j| \geq \sum_{j=1}^{\lambda_s} |Y_j|
\]

which means \(|C^s_s(Y, f^s, q_s)| \leq |C^s_s(\tilde{Y}, f^s, q_s)|\).

**Proof of Theorem 1**

This result is a corollary of the Theorem 1 of Hatfield and Kominers (2016). We assume that each school has a dynamic reserves choice rule. By Propositions 7-9 we proved that each dynamic reserve choice function has a completion that satisfies the IRC and substitutability conditions. Then, by Theorem 1 of Hatfield and Kominers (2016), there exists a stable outcome with respect to profile of schools’ choice functions.

**Proof of Theorem 3**

This result is a corollary of the Theorem 3 of Hatfield and Kominers (2016). In Proposition 9 we showed that the substitutable completion of dynamic reserves choice functions, which satisfies the IRC, satisfies the LAD condition. Then, by the Theorem 3 of Hatfield and Kominers (2016), the cumulative offer mechanism is (group) strategy proof for students.

**Proof of Theorem 4**

Fix a student \( i \in I \). Assume, towards a contradiction, that cumulative offer mechanism does not respect unambiguous improvements. Then, there exists a student \( i \in I \), a preference profile of students \( P \in \times_{i \in I} P_s^i \) and priority profiles \( \Pi \) and \( \Pi \) such that \( \Pi \) is an unambiguous improvement over \( \Pi \) for student \( i \) and

\[
\varphi_i(P; \Pi)P^i\varphi_i(P; \Pi).
\]

Let \( \varphi_i(P; \Pi) = x \) and \( \varphi_i(P; \Pi) = \bar{x} \). Consider a preference \( \tilde{P}^i \) of student \( i \) according to which only acceptable contract is \( x \), i.e., \( \tilde{P}^i : x - \emptyset_i \). Let \( \tilde{P} = (\tilde{P}^i, P_{-i}) \). We will first prove the following claim:

**Claim:** \( \varphi_i(\tilde{P}; \Pi) = x \implies \varphi_i(\tilde{P}; \Pi) = x \).
Proof of the Claim: Consider the outcome of the cumulative offer mechanism under priority profile $\Pi$ given the preference profile of students $\overline{P}$. Recall that the order of students making offers has no impact on the outcome of the cumulative offer algorithm. Therefore, we can completely ignore student $i$ and run the cumulative offer algorithm until it stops. Let $Y$ be the resulting set of contracts. At this point, student $i$ makes an offer for his only contract $x$. This might create a chain of rejections, but it does not reach student $i$. So, his contract $x$ is chosen by $s(x)$, say, by the group of seats $k$ with respect to the precedence sequence $f_{s(x)}$ of school $s(x)$. Now consider the cumulative offer process under priority profile $\overline{\Pi}$. Again, we completely ignore student $i$ and run the cumulative offer algorithm until it stops. The same outcome $Y$ is obtained because the only difference between the two cumulative offer processes is student $i$’s position in the priority rankings. At this point, student $i$ makes an offer for his only contract $x$. If $x$ is chosen by the same group of seats $k$, then the same rejection chain (if there was one in the cumulative offer process under the priority profile $\Pi$) will occur and it does not reach student $i$ because otherwise we would have a contradiction with the case under priority profile $\Pi$. The only other possibility is the following: since student $i$’s ranking is now (weakly) better under $\pi_{s(x)}$ compared to $\pi_{s(x)}$, his contract $x$ might be chosen by group of seats $l < k$. Then, it must be the case that $r_l = 0$ in the cumulative offer process under both priority profiles $\Pi$ and $\overline{\Pi}$. Therefore, by selecting $x$ the group of seats $l$ must reject some other contract. Let us call this contract $y$. If no contract of student $i(y) = j$ is chosen between groups of seats $l$ and $k$, then, by the q-responsiveness of sub-choice functions, the chosen sets of the groups of seats between $l$ and $k$ under priority profiles $\Pi$ and $\overline{\Pi}$ are the same. Hence, the number of remaining seats would be the same. In this case, $y$ is chosen in the group of seats $k$. Hence, if a rejection chain starts then it will not reach student $i$ because otherwise we could have a contradiction with the fact that $x$ was chosen at the end of the cumulative offer process under priority profile $\Pi$. A different contract of student $j$ cannot be chosen between groups of seats $l$ and $k$ because otherwise observable substitutability of dynamic reserves choice function of school $s(x)$ would be violated. Therefore, if any contract of student $j$ is chosen by groups of seats between $l$ and $k$ it must be $y$. If $y$ is chosen by a group of seat before $k$, then it must replace a contract, call it $z$. By the same reasoning, no other contract of student $i(z)$ can be chosen before group of seats $k$ because otherwise we would violate the observable substitutability of the dynamic reserve choice function of school $s(x)$. Proceeding in this fashion leads to the same contract in group of seats $k$ to be rejected.

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$^{34}$Dynamic reserves choice functions satisfy observable substitutability condition of Hatfield, Kominers and Westkamp (2017b). We refer readers to Hatfield, Kominers and Westkamp (2017b) for the definitions of observable offer processes and observable substitutability. Since dynamic reserves choice functions have substitutable completion that satisfies the size monotonicity, then it satisfies observable substitutability by Hatfield, Kominers and Westkamp (2017b).
and initiates the same rejection chain that occurs under priority profile \( \Pi \). Since the same rejection chain does not reach student \( i \) under priority profile \( \Pi \) it will not reach student \( i \) under priority profile \( \Pi \), which ends our proof for the claim.

Since \( \varphi_i(P; \Pi) = x \) and \( \varphi_i(P; \Pi) = \bar{x} \) such that \( xP^i\bar{x} \), under priority profile \( \Pi \) if student \( i \) misreports and submit \( \hat{P}^i \) then she can successfully manipulate the cumulative offer mechanism. This is a contradiction because we have already established that the cumulative offer mechanism is strategy proof.

**Proof of Theorem 5**

We prove Theorem 5 in two parts:

**Part I of the proof:** Consider school \( s \in S \) with a precedence sequence \( f^s \) and a target capacity vector \( (\tilde{q}^s_1, ..., \tilde{q}^s_{\lambda_s}) \). Let \( \tilde{q}_s \) and \( q_s \) be two capacity transfer schemes that are compatible with the precedence sequence \( f^s \) and the target capacity vector \( (\tilde{q}^s_1, ..., \tilde{q}^s_{\lambda_s}) \). Suppose that the following two conditions hold:

- there exists \( l \in \{2, ..., \lambda_s \} \) and \( (\hat{r}_1, ..., \hat{r}_{l-1}) \in \mathbb{Z}_+^{l-1} \) such that \( \tilde{q}_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) = 1 + q_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) \), and
- for all \( j \in \{2, ..., \lambda_s \} \) and \( (r_1, ..., r_{j-1}) \in \mathbb{Z}_+^{j-1} \), if \( j \neq l \) or \( (r_1, ..., r_{j-1}) \neq (\hat{r}_1, ..., \hat{r}_{l-1}) \), then \( \tilde{q}_s^j(r_1, ..., r_{j-1}) = q_s^j(r_1, ..., r_{j-1}) \).

In this part we will show that \( |C^s(Y, f^s, \tilde{q}_s)| \geq |C^s(Y, f^s, q_s)| \) for \( \tilde{q}_s \) and \( q_s \) defined above.

Take an offer set \( Y \subseteq X \). Note that each contract \( x \in Y \setminus Y_s \) in which \( s \) is not a party is irrelevant. By the IRC condition of dynamic reserve choice functions we focus only on \( Y_s \). In the computation of \( \tilde{C}^s(Y_s, f^s, \tilde{q}_s) \), if the vector of number of vacant slots \( (\hat{r}_1, ..., \hat{r}_{l-1}) \in \mathbb{Z}_+^{l-1} \) is not realized, then \( \tilde{C}^s(Y_s, f^s, \tilde{q}_s) = C^s(Y_s, f^s, q_s) \) because two capacity transfer schemes differ only at that vector. Hence, we have that \( |\tilde{C}^s(Y_s, f^s, \tilde{q}_s)| = |C^s(Y_s, f^s, q_s)| \).

Now suppose that the vector of number of vacant slots \( (\hat{r}_1, ..., \hat{r}_{l-1}) \in \mathbb{Z}_+^{l-1} \) is realized in the computation of \( \tilde{C}^s(Y_s, f^s, \tilde{q}_s) \). Let \( (Y_j, r_j, Y^j) \) and \( (\tilde{Y}_j, \tilde{r}_j, \tilde{Y}^j) \) be the sequences of sets of chosen contracts, numbers of vacant seats and sets of remaining contracts for group of seats \( j = 1, ..., \lambda_s \) under choice processes beginning with \( Y_s \) using capacity transfer schemes \( q_s \) and \( \tilde{q}_s \), respectively. Since the two choice processes start to differ exactly at the group of seat \( l \), we have \( Y_j = \tilde{Y}_j \), \( r_j = \tilde{r}_j \), and \( Y^j = \tilde{Y}^j \) for all \( j = 1, ..., l-1 \). We know that \( \tilde{q}_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) = 1 + q_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) \). In group of seats \( l \) there are two cases:
1. \( \tilde{Y}_i = \{x\} \cup Y_i \) for some \( x \). Then, in this case, we have \( r_i = \tilde{r}_i = 0 \) since the sub-choice function is \( q \)-responsive. Hence, by the definition of \( \tilde{q}_s \), we have

\[
\tilde{q}_s^{i+1}(\tilde{r}_1, \ldots, \tilde{r}_{i-1}, 0) = q_s^{i+1}(\tilde{r}_1, \ldots, \tilde{r}_{i-1}, 0).
\]

Moreover, since under \( \tilde{q}_s \) an extra contract \( x \) is chosen compared to the chosen set under \( q \), we have

\[
i \left( \bigcup_{j=1}^{i} Y_j \right) \subset i \left( \bigcup_{j=1}^{i} \tilde{Y}_j \right) \quad \text{and} \quad |i \left( \bigcup_{j=1}^{i} Y_j \right)| + 1 = |i \left( \bigcup_{j=1}^{i} \tilde{Y}_j \right)|.
\]

2. \( \tilde{Y}_i = Y_i \). In this case, \( \tilde{r}_i = 1 + r_i \). Notice that \( \tilde{q}_s \) and \( q_s \) differs only at the vector \((\tilde{r}_1, \ldots, \tilde{r}_{i-1}) \in \mathbb{Z}_+^{i-1}\). We have \((\tilde{r}_1, \ldots, \tilde{r}_{i-1}, r_i) \) under \( q_s \) and \((\hat{r}_1, \ldots, \hat{r}_{i-1}, 1+r_i) \) under \( \tilde{q}_s \). By the monotonicity of the capacity transfer schemes \(^{35}\)

\[
\tilde{q}_s^{i+1}(\hat{r}_1, \ldots, \hat{r}_{i-1}, 1+r_i) = q_s^{i+1}(\tilde{r}_1, \ldots, \tilde{r}_{i-1}, 1+r_i) \in \{q_s^{i+1}(\hat{r}_1, \ldots, \hat{r}_{i-1}, r_i), 1 + q_s^{i+1}(\hat{r}_1, \ldots, \hat{r}_{i-1}, r_i)\}.
\]

In this case, we have that

\[
i \left( \bigcup_{j=1}^{i} Y_j \right) = i \left( \bigcup_{j=1}^{i} \tilde{Y}_j \right).
\]

Our argument proceeds by induction with the following hypothesis:

For all \( \tau \geq l \) either (1) or (2) holds:

1. \( i \left( \bigcup_{j=1}^{\tau} Y_j \right) \subset i \left( \bigcup_{j=1}^{\tau} \tilde{Y}_j \right) \),

\[
|\left( \bigcup_{j=1}^{\tau} Y_j \right)| + 1 = |\left( \bigcup_{j=1}^{\tau} \tilde{Y}_j \right)|, \quad \text{and} \quad q_s^{1+\tau}(r_1, \ldots, r_\tau) = \tilde{q}_s^{1+\tau}(\tilde{r}_1, \ldots, \tilde{r}_\tau).
\]

2. \( i \left( \bigcup_{j=1}^{\tau} Y_j \right) = i \left( \bigcup_{j=1}^{\tau} \tilde{Y}_j \right) \) and

\[
q_s^{1+\tau}(r_1, \ldots, r_\tau) \in \{q_s^{1+\tau}(r_1, \ldots, r_\tau), 1 + q_s^{1+\tau}(r_1, \ldots, r_\tau)\}.
\]

We have already showed that it holds for \( \tau = l \). Now, assume that either (1) or (2) holds for all \( \tau = l, \ldots, k - 1 \). We need to show that either (1) or (2) holds for group of seats \( k \).

**Case 1:** First suppose that (1) holds for group of seats \( \tau = k - 1 \). Then, by our inductive assumption, we have \( q_s^k = \tilde{q}_s^k \). Our inductive assumption in this case also implies that \( \tilde{Y}^{k-1} \subset Y^{k-1} \). Moreover the difference \( Y^{k-1} \setminus \tilde{Y}^{k-1} \) is a set of contracts that the same students is a party of, i.e., \( |i(\tilde{Y}^j \setminus \tilde{Y}^j)| = 1 \). Let us call this student \( \tilde{i} \).

There are two possibility: (a) \( \tilde{i} \notin i(Y_k) \) and (b) \( \tilde{i} \in i(Y_k) \).

In the case of (a), \( Y_k = \tilde{Y}_k \) by the IRC of the sub-choice function. Therefore, \( r_k = \tilde{r}_k \) since \( q_s^k = \tilde{q}_s^k \). Then, we will have

\[
i \left( \bigcup_{j=1}^{k} Y_j \right) \subset i \left( \bigcup_{j=1}^{k} \tilde{Y}_j \right) \quad \text{and} \quad |\left( \bigcup_{j=1}^{k} Y_j \right)| + 1 = |\left( \bigcup_{j=1}^{k} \tilde{Y}_j \right)|.
\]

\(^{35}\)By the first condition of the monotonicity we have \( q_s^{i+1}(\hat{r}_1, \ldots, \hat{r}_{i-1}, 1 + r_i) \geq q_s^{i+1}(\hat{r}_1, \ldots, \hat{r}_{i-1}, r_i) \) but the difference between them has an upper bound (which is 1) by the second condition of the monotonicity, i.e., \( q_s^{i+1}(\hat{r}_1, \ldots, \hat{r}_{i-1}, 1 + r_i) - q_s^{i+1}(\hat{r}_1, \ldots, \hat{r}_{i-1}, r_i) \leq 1 \).
By the definitions of transfers. By the second condition of the monotonicity of capacity transfers we have
\[ q^{k+1}_s(\bar{r}_1, ..., \bar{r}_k) \geq q^{k+1}_s(r_1, ..., r_k) \] by the first condition of the monotonicity of capacity transfers. By the second condition of the monotonicity of capacity transfers we have
\[ \sum_{j=1}^{k+1} [q^j_s(\bar{r}_1, ..., \bar{r}_{j-1}) - q^j_s(r_1, ..., r_{j-1})] \leq \sum_{j=1}^{k} [\bar{r}_j - r_j]. \]

By the definitions of \( r \) and \( \bar{r} \) we have
\[ \sum_{j=1}^{k} [\bar{r}_j - r_j] = \sum_{j=1}^{k} [q^j_s(\bar{r}_1, ..., \bar{r}_{j-1}) - |\tilde{Y}_j|] - q^j_s(r_1, ..., r_{j-1}) + |Y_j|. \]

By the construction of \( \tilde{q}_s \), we have
\[ \sum_{j=1}^{k} q^j_s(\bar{r}_1, ..., \bar{r}_{j-1}) = \left[ \sum_{j=1, j \neq \ell}^{k} q^j_s(\bar{r}_1, ..., \bar{r}_{j-1}) \right] + q^\ell_s(\bar{r}_1, ..., \bar{r}_{\ell-1}) + 1. \]

The last equality implies
\[ \sum_{j=1}^{k} [\bar{r}_j - r_j] = \sum_{j=1}^{k} [q^j_s(\bar{r}_1, ..., \bar{r}_{j-1}) - q^j_s(r_1, ..., r_{j-1})] + \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) + 1 \]
where \( \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) = -1 \) by our inductive assumption the fact that \( |Y_k| = |\tilde{Y}_k| \).

Hence, we have
\[ \sum_{j=1}^{k} [\bar{r}_j - r_j] = \sum_{j=1}^{k} [q^j_s(\bar{r}_1, ..., \bar{r}_{j-1}) - q^j_s(r_1, ..., r_{j-1})]. \]

Plugging this last equality in
\[ \sum_{j=1}^{k+1} [q^j_s(\bar{r}_1, ..., \bar{r}_{j-1}) - q^j_s(r_1, ..., r_{j-1})] \leq \sum_{j=1}^{k} \bar{r}_j - r_j \]
will give us
\[ q^{k+1}_s(\bar{r}_1, ..., \bar{r}_k) - q^{k+1}_s(r_1, ..., r_k) \leq 0 \]
which means \( q^{k+1}_s(\bar{r}_1, ..., \bar{r}_k) \leq q^{k+1}_s(r_1, ..., r_k) \). Therefore, we must have \( q^{k+1}_s(\bar{r}_1, ..., \bar{r}_k) = q^{k+1}_s(r_1, ..., r_k). \)

In the case of \( (b) \), we assume a contract of \( \tilde{i} \), call it \( \tilde{x} \), is chosen in \( Y_k \). If \( |\tilde{Y}_k| < q^k_s(\bar{r}_1, ..., \bar{r}_{k-1}) \) then we have \( Y_k = \tilde{Y}_k \cup \{\tilde{x}\} \) since the sub-choice function is \( q \)-responsive. In this case, we will have \( i \left[ \bigcup_{j=1}^{k} Y_j \right] = i \left[ \bigcup_{j=1}^{k} \tilde{Y}_j \right] \). In this case we will have \( \bar{r}_k = 1 + r_k \).

\(^{36}\)We will use this equality repeatedly for the rest of the proof.
Again, monotonicity of capacity transfers gives us

$$q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) = q_{k+1}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \geq q_{s}^{k+1}(r_1, ..., r_k)$$

by the first condition of the monotonicity definition. We again invoke the following equality here:

$$\sum_{j=1}^{k} [\tilde{r}_j - r_j] = \sum_{j=1}^{k} [q_{s}^{k}(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_{s}^{k}(r_1, ..., r_{j-1})] + \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) + 1$$

where \( \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) = 0 \) by our inductive assumption and the fact that \(|Y_k| = |\tilde{Y}_k| + 1\). Plugging it in the second condition of monotonicity of capacity transfers gives us

$$q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) = q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) - q_{s}^{k+1}(r_1, ..., r_k) \leq 1$$

Hence, we will have \( q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \in \{ q_{s}^{k+1}(r_1, ..., r_k), 1 + q_{s}^{k+1}(r_1, ..., r_k) \} \).

If \(|\tilde{Y}_k| = q_{s}^{k}(\tilde{r}_1, ..., \tilde{r}_{k-1})\) then we have \( Y_k = \tilde{Y}_k \cup \{ \tilde{z} \} \) for some contract \( \tilde{z} \). In this case, \( r_k = \tilde{r}_k = 0 \). We will have \( i \left( \bigcup_{j=1}^{k} Y_j \right) \subset i \left( \bigcup_{j=1}^{k} \tilde{Y}_j \right) \), \( |i \left( \bigcup_{j=1}^{k} Y_j \right)| + 1 = |i \left( \bigcup_{j=1}^{k} \tilde{Y}_j \right)| \).

Since we have \( \tilde{r}_j \geq r_j \) for all \( j = 1, ..., k - 1 \) we have \( q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) = q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \geq q_{s}^{k+1}(r_1, ..., r_k) \) by the first condition of the monotonicity of capacity transfers. We again invoke the following equality here:

$$\sum_{j=1}^{k} [\tilde{r}_j - r_j] = \sum_{j=1}^{k} [q_{s}^{k}(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_{s}^{k}(r_1, ..., r_{j-1})] + \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) + 1$$

where \( \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) = -1 \) by our inductive assumption and the fact that \(|Y_k| = |\tilde{Y}_k| \). Hence, we have

$$\sum_{j=1}^{k} [\tilde{r}_j - r_j] = \sum_{j=1}^{k} [q_{s}^{k}(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_{s}^{k}(r_1, ..., r_{j-1})]$$

Plugging this last equality in \( \sum_{j=1}^{k+1} [q_{s}^{j}(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_{s}^{j}(r_1, ..., r_{j-1})] \leq \sum_{j=1}^{k} [\tilde{r}_j - r_j] \) will give us

$$q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) - q_{s}^{k+1}(r_1, ..., r_k) \leq 0$$

which means \( q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \leq q_{s}^{k+1}(r_1, ..., r_k) \). Therefore, we must have \( q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) = \tilde{q}_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) = q_{s}^{k+1}(r_1, ..., r_k) \).
Case 2: Now suppose that (2) holds for group of seats \( \tau = k - 1 \). In this case, we have \( Y^{k-1} = \tilde{Y}^{k-1} \). If we have \( \tilde{q}_s^k(\tilde{r}_1, ..., \tilde{r}_{k-1}) = q^k(r_1, ..., r_{k-1}) \) then we will have \( Y_k = \tilde{Y}_k \) and \( r_k = \tilde{r}_k \). Hence, \( i \left[ \bigcup_{j=1}^{k} Y_j \right] = i \left[ \bigcup_{j=1}^{k} \tilde{Y}_j \right] \). Moreover, by monotonicity of capacity transfers we will have \( q^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) = q^{k+1}(r_1, ..., r_k) \). Because, by our inductive assumption, \( \tilde{r}_j \geq r_j \) for all \( j = 1, ..., k-1 \). It implies \( q^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) = q^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \geq q^{k+1}(r_1, ..., r_k) \) by the first condition of monotonicity of capacity transfers. We again use the following equality:

\[
\sum_{j=1}^{k} \tilde{r}_j - r_j = \sum_{j=1}^{k} [q^k_s(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q^k_s(r_1, ..., r_{j-1})] + \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) + 1
\]

where \( \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) = 0 \) by our inductive assumption and the fact that \( |Y_k| = |\tilde{Y}_k| \). Hence, we have

\[
\sum_{j=1}^{k+1} \tilde{r}_j - r_j = 1 + \sum_{j=1}^{k} [q^k_s(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q^k_s(r_1, ..., r_{j-1})] .
\]

Plugging this last equality in \( \sum_{j=1}^{k+1} [q^k_s(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q^k_s(r_1, ..., r_{j-1})] \leq \sum_{j=1}^{k} \tilde{r}_j - r_j \) will give us

\[
q^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) - q^{k+1}(r_1, ..., r_k) \leq 1.
\]

Therefore, we have \( q^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) = \tilde{q}^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) \in \{q^{k+1}_s(r_1, ..., r_k), 1+q^{k+1}_s(r_1, ..., r_k)\} \).

Now suppose that we have \( i \left[ \bigcup_{j=1}^{k-1} Y_j \right] = i \left[ \bigcup_{j=1}^{k-1} \tilde{Y}_j \right] \) and \( \tilde{q}^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_{k-1}) = 1+q^{k+1}_s(r_1, ..., r_{k-1}) \) there are two sub-cases. The first one is where \( |\tilde{Y}_k| = \tilde{q}^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_{k-1}) \). This implies \( |Y_k| = q^{k+1}_s(r_1, ..., r_{k-1}) \) which, in turn, implies \( r_k = \tilde{r}_k = 0 \), because the sub-choice function is q-responsive. Hence, we will have \( i \left[ \bigcup_{j=1}^{k} Y_j \right] \subset i \left[ \bigcup_{j=1}^{k} \tilde{Y}_j \right] , |i \left[ \bigcup_{j=1}^{k} Y_j \right] | + 1 = |i \left[ \bigcup_{j=1}^{k} \tilde{Y}_j \right]| \). With regard to the capacity of the next group, given that \( \tilde{r}_j \geq r_j \) for all \( j = 1, ..., k \), the first condition of monotonicity implies

\[
\tilde{q}^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) = q^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) \geq q^{k+1}_s(r_1, ..., r_k) .
\]

We invoke again the following equality

\[
\sum_{j=1}^{k} \tilde{r}_j - r_j = \sum_{j=1}^{k} [q^k_s(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q^k_s(r_1, ..., r_{j-1})] + \sum_{j=1}^{k} \left( |Y_j| - |\tilde{Y}_j| \right) + 1
\]
where $\sum_{j=1}^{k} (|Y_j| - |\tilde{Y}_j|) = -1$ given the inductive assumption and the fact that $|\tilde{Y}_k| = 1 + |Y_k|$. Hence, we have

$$\sum_{j=1}^{k} (\tilde{r}_j - r_j) = \sum_{j=1}^{k} \left[q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_s^j(r_1, ..., r_{j-1})\right].$$

Plugging it into $\sum_{j=1}^{k} [q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_s^j(r_1, ..., r_{j-1})] \leq \sum_{j=1}^{k} \tilde{r}_j - r_j$ will give us

$$q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) - q_{s}^{k+1}(r_1, ..., r_k) \leq 0.$$ 

Hence, we conclude that

$$\tilde{q}_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) = q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) = q_{s}^{k+1}(r_1, ..., r_k).$$

Finally, suppose that we have $i \left[\bigcup_{j=1}^{k-1} Y_j\right] = i \left[\bigcup_{j=1}^{k-1} \tilde{Y}_j\right]$ and $\tilde{q}_{s}^{k}(\tilde{r}_1, ..., \tilde{r}_{k-1}) = 1 + q_{s}^{k}(r_1, ..., r_{k-1})$ and $|\tilde{Y}_k| < \tilde{q}_{s}^{k}(\tilde{r}_1, ..., \tilde{r}_{k-1})$. In this case, we will have $|\tilde{Y}_k| = |Y_k|$ because $\tilde{Y}_{k-1} = Y_{k-1}$ and the sub-choice function is $q$-responsive. This, in turn, implies that $\tilde{r}_k = 1 + r_k$. Again, given that $\tilde{r}_j \geq r_j$ for all $j = 1, ..., k$, the first condition of monotonicity implies

$$\tilde{q}_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) = q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) \geq q_{s}^{k+1}(r_1, ..., r_k).$$

We invoke following equality yet again

$$\sum_{j=1}^{k} (\tilde{r}_j - r_j) = \sum_{j=1}^{k} \left[q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_s^j(r_1, ..., r_{j-1})\right] + \sum_{j=1}^{k} (|Y_j| - |\tilde{Y}_j|) + 1$$

where $\sum_{j=1}^{k} (|Y_j| - |\tilde{Y}_j|) = 0$ given the inductive assumption and the fact that $|\tilde{Y}_k| = |Y_k|$. Hence, we have

$$\sum_{j=1}^{k} (\tilde{r}_j - r_j) = 1 + \sum_{j=1}^{k} \left[q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_s^j(r_1, ..., r_{j-1})\right].$$

Plugging it into $\sum_{j=1}^{k} [q_s^j(\tilde{r}_1, ..., \tilde{r}_{j-1}) - q_s^j(r_1, ..., r_{j-1})] \leq \sum_{j=1}^{k} \tilde{r}_j - r_j$ will give us

$$q_{s}^{k+1}(\tilde{r}_1, ..., \tilde{r}_k) - q_{s}^{k+1}(r_1, ..., r_k) \leq 1.$$
Hence, we conclude that

\[ \tilde{q}^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) = q^{k+1}_s(\tilde{r}_1, ..., \tilde{r}_k) \in \{q^{k+1}_s(r_1, ..., r_k), 1 + q^{k+1}_s(r_1, ..., r_k)\}. \]

We proved that for each group of seats \( \tau \geq l \) either \( i \left[ \bigcup_{j=1}^{\tau} Y_j \right] | +1 \geq i \left[ \bigcup_{j=1}^{\tau} \tilde{Y}_j \right] \) or \( i \left[ \bigcup_{j=1}^{\tau} Y_j \right] = i \left[ \bigcup_{j=1}^{\tau} \tilde{Y}_j \right] \) holds. Particularly, it holds for \( \tau = \lambda_s \). Thus, we have \( |C^s(Y, f^s, \tilde{q}_s)| \geq |C^s(Y, f^s, q_s)| \).

**Part II of the proof:** Let \( q_s \) and \( \tilde{q}_s \) be two monotone capacity transfer schemes that are compatible with the precedence sequence \( f^s \) and target capacity vector \( \bar{q}_s \) of school \( s \) and such that \( \tilde{q}_s \) is more flexible than \( q_s \). Then, by definition,

1. there exists \( l \in \{2, ..., \lambda_s\} \) and \( (\hat{r}_1, ..., \hat{r}_{l-1}) \in \mathbb{Z}^{l-1}_+ \) such that \( \tilde{q}_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) > q_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) \), and

2. for all \( j \in \{2, ..., \lambda_s\} \) and \( (r_1, ..., r_{j-1}) \in \mathbb{Z}^{j-1}_+ \), if \( j \neq l \) or \( (r_1, ..., r_{j-1}) \neq (\hat{r}, ..., \hat{r}_{l-1}) \), then \( \tilde{q}_s^l(r_1, ..., r_{j-1}) \geq q_s^l(r_1, ..., r_{j-1}) \).

We now define a sequence of monotone capacity transfer schemes so that the sequence starts with \( q_s \) and ends with \( \tilde{q}_s \) and two consequent monotone capacity transfer schemes differ only at a unique vector of number of unused seats as defined in the part I of the proof. Let \( A = \{(\bar{r}_1, ..., \bar{r}_{l-1}) \in \mathbb{Z}^{l-1}_+ | \tilde{q}_s^l(\bar{r}_1, ..., \bar{r}_{l-1}) > q_s^l(\bar{r}_1, ..., \bar{r}_{l-1})\} \) and \( \kappa: \mathbb{Z}_+ \rightarrow A \) be a surjective function such that

- for each pairs of vectors \( (r_1, ..., r_{l-1}), (\hat{r}_1, ..., \hat{r}_{m-1}) \in A \), we have \( \kappa^{-1}((r_1, ..., r_{l-1})) < \kappa^{-1}((\hat{r}_1, ..., \hat{r}_{m-1})) \) if \( 2 \leq l < m \),

- for each pairs of vectors \( (r_1, ..., r_{l-1}), (\hat{r}_1, ..., \hat{r}_{l-1}) \in A \), we have \( \kappa^{-1}((r_1, ..., r_{l-1})) < \kappa^{-1}((\hat{r}_1, ..., \hat{r}_{l-1})) \) if either \( r_1 < \hat{r}_1 \) or \( [r_1 = \hat{r}_1 \text{ and } r_2 < \hat{r}_2] \) or \([r_1 = \hat{r}_1, r_2 = \hat{r}_2 \text{ and } r_3 < \hat{r}_3] \), or \([r_1 = \hat{r}_1, ..., r_{l-2} = \hat{r}_{l-2} \text{ and } r_{l-1} < \hat{r}_{l-1}] \).

For any given set of contracts \( Y \subseteq X \), consider the following sequence of monotone capacity transfer schemes and chosen sets

\[ ((q_s^1, (q_s)^2, ...) \text{ and } (C^s(Y, f^s, (q_s)^1), C^s(Y, f^s, (q_s)^2), ...) \]

such that \( (q_s)^1 = q_s \) and for any \( p > 1 \)

- \( (q_s^p)^p(\kappa(p)) = 1 + (q_s^p)^p(\kappa(p)) \) if \( \kappa(p) = (\hat{r}_1, ..., \hat{r}_{l-1}) \in A \), and
• \((q_s^n)^p(r_1, ..., r_{m-1}) = (q_s^n)^{p-1}(r_1, ..., r_{m-1})\) if \(m \neq l\) or \((r_1, ..., r_{m-1}) \neq \kappa(p)\).

As shown in the part I of the proof, given \(Y \subseteq X\), for every \((q_s)^p\) and \((q_s)^{p-1}\) we have \(|C_s(Y, f^s, (q_s)^p)|\geq|C_s(Y, f^s, (q_s)^{p-1})|\). Since we have finitely many groups of seats in school \(s\) and finitely many seats in each group, the sequence described above reaches the monotone capacity transfer scheme \(\bar{q}_s\) from \(q_s\) in finitely many steps. Since the number of chosen contracts weakly increase at each step we conclude \(|C_s(Y, f^s, \bar{q}_s)| \geq|C_s(Y, f^s, q_s)|\).

**Proof of Theorem 6**

Consider school \(s \in S\) with a precedence sequence \(f^s\) and a target capacity vector \((\bar{q}_s^1, ..., \bar{q}_s^r)^s\). Let \(\bar{q}_s\) and \(q_s\) be two capacity transfer schemes that are compatible with the precedence sequence \(f^s\) and the target capacity vector \((\bar{q}_s^1, ..., \bar{q}_s^r)^s\). Suppose that the following two conditions hold:

• there exists \(l \in \{2, ..., \lambda_s\}\) and \((\hat{r}_1, ..., \hat{r}_{l-1}) \in \mathbb{Z}_{+}^{l-1}\) such that \(\bar{q}_s^l(\hat{r}_1, ..., \hat{r}_{l-1}) = 1 + q_s^l(\hat{r}_1, ..., \hat{r}_{l-1})\), and

• for all \(j \in \{2, ..., \lambda_s\}\) and \((r_1, ..., r_{j-1}) \in \mathbb{Z}_{+}^{j-1}\), if \(j \neq l\) or \((r_1, ..., r_{j-1}) \neq (\hat{r}_1, ..., \hat{r}_{l-1})\), then \(\bar{q}_s^j(r_1, ..., r_{j-1}) = q_s^j(r_1, ..., r_{j-1})\).

Let \(\tilde{C}^s\) and \(C^s\) be dynamic reserves choice functions \(\tilde{C}^s(\cdot, f^s, \bar{q}_s)\) and \(C^s(\cdot, f^s, q_s)\), respectively. Let \(\tilde{C} = (\tilde{C}^s, C_{-s})\) and \(C = (C^s, C_{-s})\). Let the outcomes of the cumulative offer algorithm at \((P, \tilde{C})\) and \((P, C)\) be \(\tilde{Z}\) and \(Z\), respectively. If \(\tilde{Z} = Z\), then there is nothing to prove because it means capacity flexibility of school \(s\) does not bite.

Suppose that \(\tilde{Z} \neq Z\). That is, capacity flexibility of school \(s\) bites, which means that there is a student who was rejected under \(C^s\) is no longer rejected under \(\tilde{C}^s\). We now define an *improvement chains* algorithm that starts with the outcome \(Z\). Since the capacity flexibility bites, the vector \((\hat{r}_1, ..., \hat{r}_{l-1})\) must occur in the choice procedure of school \(s\).

**Step 1:** Consider students who prefer \((s, f^s(l))\) to their assignments under \(Z\), i.e.,

\[\tilde{I}_1^{(s, f^s(l))} = \{i \in I \mid (s, f^s(l))P^s Z_i\}.\]

We choose \(\pi^s\)-maximal student in \(\tilde{I}_1^{(s, f^s(l))}\) (if any), call her \(\tilde{i}_1\), and assign her \(\tilde{x}_1 = (\tilde{i}_1, s, f^s(l))\). Update the outcome to \(\tilde{Z}_1 = Z \cup \{\tilde{x}_1\} \setminus z_1\) where \(z_1\) is the contract student \(\tilde{i}_1\) receives under \(Z\).
If \((s(z_1), t(z_1)) = \emptyset\), then the improvement process ends and we have \(\tilde{Z} = \tilde{Z}_1 = Z \cup \{\tilde{x}_1\}\). Otherwise, we move to Step 2 because by assigning \(\tilde{i}_1\) to \((s, f^s(l))\) we create a vacancy in school \(s(z_1)\) within the privilege \(t(z_1)\).

If \(\tilde{I}^{(s,f^s(l))}_1 = \emptyset\), then the number of vacant seats at the last group of seats accepting students in type \(f^s(l)\) will increase by one. If the capacity transfer scheme of school \(s\) does not transfer this extra vacancy to any other group following the last group of seats in type \(f^s(l)\) in the computation of \(C^s(Z_s, f^s, \bar{q}_s)\), then the improvement chain process ends and we have \(\tilde{Z} = Z\). If the extra seat is transferred to the group of seat \(l'\) which follows the last group of seats in type \(f^s(l)\) in the computation of \(C^s(Z_s, f^s, \bar{q}_s)\), then we consider students who prefers \((s, f^s(l'))\) over their assignments under \(Z\), i.e.,

\[
I^{(s,f^s(l'))}_1 = \{i \in I \mid (s, f^s(l')) P^s Z_i\}.
\]

We choose \(\pi^s\)-maximal student in \(I^{(s,f^s(l'))}_1\) (if any), call her \(\tilde{i}_1\), and assign her \(\tilde{x}_1 = (\tilde{i}_1, s, f^s(l'))\). Update the outcome to \(\tilde{Z}_1 = Z \cup \{\tilde{x}_1\} \setminus z_1\) where \(z_1\) is the contract \(\tilde{i}_1\) receives under \(Z\).

If \((s(z_1), t(z_1)) = \emptyset\), then the improvement process ends and we have \(\tilde{Z} = \tilde{Z}_1 = Z \cup \{\tilde{x}_1\}\). Otherwise, we move to Step 2 because by assigning \(\tilde{i}_1\) to \((s, f^s(l'))\) we create a vacancy in school \(s(z_1)\) within the privilege \(t(z_1)\).

If \(\tilde{I}^{(s,f^s(l'))}_1 = \emptyset\), then the number of vacant seats at the last group of seats accepting students in type \(f^s(l')\) will increase by one. If the capacity transfer scheme of school \(s\) does not transfer this extra vacancy to any other group following the last group of seats in type \(f^s(l')\) in the computation of \(C^s(Z_s, f^s, \bar{q}_s)\), then the improvement chain process ends and we have \(\tilde{Z} = Z\). If the extra seat is transferred to the group of seat \(l''\) which follows the last group of seats in type \(f^s(l')\) in the computation of \(C^s(Z_s, f^s, \bar{q}_s)\), then we consider students who prefers \((s, f^s(l''))\) over their assignments under \(Z\), and so on.

Since school \(s\) has finitely many groups of seats, Step 1 ends in finitely many iterations. If no extra student is assigned to school \(s\) by the end of Step 1, then the improvement chains algorithm ends and we have \(\tilde{Z} = Z\). If an extra student is assigned to school \(s\) by the end of Step 1, then we move on to Step 2.

**Step t > 1:** Consider students who prefer \((s(z_{t-1}), t(z_{t-1}))\) to their assignments under \(\tilde{Z}_{t-1}\), i.e.,

\[
\tilde{I}^{(s(z_{t-1}), t(z_{t-1}))}_t = \{i \in I \mid (s(z_{t-1}), t(z_{t-1})) P^s (\tilde{Z}_{t-1}) i\}.
\]

We choose \(\pi^s(z_{t-1})\)-maximal student in \(\tilde{I}^{(s(z_{t-1}), t(z_{t-1}))}_t\), call her \(\tilde{i}_t\), and assign her \(\tilde{x}_t = (\tilde{i}_t, s(z_{t-1}), t(z_{t-1}))\). Update the outcome to \(\tilde{Z}_t = \tilde{Z}_{t-1} \cup \{\tilde{x}_t\} \setminus z_t\) where \(z_t\) is the contract student \(\tilde{i}_t\) receives under \(\tilde{Z}_{t-1}\).

If \((s(z_{t-1}), t(z_{t-1})) = \emptyset\), then the improvement algorithm ends and we have \(\tilde{Z} = \tilde{Z}_t = \tilde{Z}_{t-1}\).
\( \tilde{Z}_{t-1} \cup \{ \tilde{x}_t \} \). Otherwise, we move to Step \( t + 1 \) because by assigning \( \tilde{i}_t \) to \( (s(z_{t-1}), t(z_{t-1})) \) we create a vacancy in school \( s(z_t) \) within type \( t(z_t) \).

If \( \tilde{I}_t(s(z_{t-1}), t(z_{t-1})) = \emptyset \), then the number of vacant seats at the last group of seats accepting students in type \( f^{s(z_{t-1})} \) will increase by one. If the capacity transfer scheme of school \( s(z_{t-1}) \) does not transfer this extra capacity to any other group following the last group of seats in type \( t(z_{t-1}) \) in the computation of \( C^{s(z_{t-1})}((\tilde{Z}_{t-1})_{s(z_{t-1})}, f^{s(z_{t-1})}, q_{s(z_{t-1})}) \), then the improvement chains process ends we have \( \tilde{Z} = \tilde{Z}_{t-1} \). If the extra seat is transferred to the group of seat \( m \) which follows the last group of seats in type \( t(z_{t-1}) \) in the computation of \( C^{s(z_{t-1})}((\tilde{Z}_{t-1})_{s(z_{t-1})}, f^{s(z_{t-1})}, q_{s(z_{t-1})}) \), then we consider students who prefers \( (s(z_{t-1}), f^{s(z_{t-1})}(m)) \) over their assignments under \( \tilde{Z}_{t-1} \), i.e.,

\[
\tilde{I}_t^{s(z_{t-1}), f^{s(z_{t-1})}(m)} = \{ i \in I | (s(z_{t-1}), f^{s(z_{t-1})}(m)) \in P^i(\tilde{Z}_{t-1}) \}.
\]

We choose \( \pi^{s(z_{t-1})} \)-maximal student in \( \tilde{I}_t^{s(z_{t-1}), f^{s(z_{t-1})}(m)} \), call her \( \tilde{i}_t \), and assign her \( \tilde{x}_t = (\tilde{i}_t, s(z_{t-1}), f^{s(z_{t-1})}(m)) \). Update the outcome to \( \tilde{Z} = \tilde{Z}_{t-1} \cup \{ \tilde{x}_t \} \setminus z_t \) where \( z_t \) is the contract student \( \tilde{i}_t \) receives under \( \tilde{Z}_{t-1} \).

If \( (s(z_{t-1}), f^{s(z_{t-1})}(m)) = \emptyset \), then the improvement algorithm ends and we have \( \tilde{Z} = \tilde{Z}_{t-1} \cup \{ \tilde{x}_t \} \). Otherwise, we move to Step \( t + 1 \) because by assigning \( \tilde{i}_t \) to \( (s(z_{t-1}), f^{s(z_{t-1})}(m)) \) we create a vacancy in school \( s(z_t) \) within type \( t(z_t) \).

If \( \tilde{I}_t^{s(z_{t-1}), t(z_{t-1})) = \emptyset \), then the number of vacant seats at the last group of seats accepting students in type \( f^{s(z_{t-1})} \) will increase by one. If the capacity transfer scheme of school \( s(z_{t-1}) \) does not transfer this extra capacity to any other group following the last group of seats in type \( f^{s(z_{t-1})}(m) \) in the computation of \( C^{s(z_{t-1})}((\tilde{Z}_{t-1})_{s(z_{t-1})}, f^{s(z_{t-1})}, q_{s(z_{t-1})}) \), then the improvement chains process ends we have \( \tilde{Z} = \tilde{Z}_{t-1} \). If the extra seat is transferred to the group of seat \( m' \) which follows the last group of seats in type \( f^{s(z_{t-1})}(m) \) in the computation of \( C^{s(z_{t-1})}((\tilde{Z}_{t-1})_{s(z_{t-1})}, f^{s(z_{t-1})}, q_{s(z_{t-1})}) \), then we consider students who prefers \( (s(z_{t-1}), f^{s(z_{t-1})}(m')) \) over their assignments under \( \tilde{Z}_{t-1} \), and so on.

Since school \( s(z_{t-1}) \) has finitely group of seats, Step \( t \) ends in finitely many iterations. If no extra student is assigned to school \( s(z_{t-1}) \) by the end of Step \( t \), then the improvement chains algorithm ends and we have \( \tilde{Z} = \tilde{Z}_{t-1} \). If an extra student is assigned to school \( s(z_{t-1}) \) by the end of the Step \( t \), then we move on to Step \( t + 1 \).

This process ends in finitely many times because there are finitely many contracts and when we move to the next step it means a student is made strictly better off. Also, notice that no student gets worse of during the execution of the improvement chains algorithm. The improvement algorithm, by construction, starts with the outcome \( \Phi(P, \bar{C}) \) and ends at \( \Phi(P, \tilde{C}) \). Hence, we have \( \Phi_i(P, \tilde{C})R^i\Phi_i(P, \bar{C}) \) for all \( i \in I \).

We define the same sequence of capacity transfer schemes and dynamic reserve choice functions for school \( s \in S \) as in the part II of the proof of Theorem 5: \( ((q_s)^1, (q_s)^2, ...) \)
and \((C_s(Y, f^s, (q_s)^1)), C_s(Y, f^s, (q_s)^2), \ldots\). Let the sequence \(\Phi(P, C^1), \Phi(P, C^2), \ldots\) denote the outcomes of the cumulative offer processes at profiles \((P, (C_s(\cdot, f^s, (q_s)^1), C_{-s})), (P, (C_s(\cdot, f^s, (q_s)^2), C_{-s})), \ldots\), respectively. Hence, by construction, we have \(\Phi_i(P, C^{a+1}) \sim^i \Phi_i(P, C^a)\) for all \(i \in I\) and \(a \geq 1\). By transitivity of weak preferences, we have \(\Phi_i(P, \tilde{C}) \sim^i \Phi(P, C)\) for all \(i \in I\).

References


