The Limit of Targeting in Networks*

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Abstract

Network-based targeting is valuable in many applications such as diffusion of new technology, product promotion in marketing, crowd sourcing among others. Nevertheless, how to quantitatively measure the effectiveness of targeting strategies remains a challenge. This paper studies a class of network games with strategic complements, where a designer can choose finite sequences of targeting interventions. We propose an effectiveness index, called *relative network synergy equivalent* (RNSE), to measure the effect of such network-based targeting interventions. Regardless of the targeting policies and the network structures, a simple upper bound for this index is $\sqrt{2} \approx 1.414$. This unified bound is applicable with multiple stages of moves, revenue maximization, costly seeding, and random seeding as the alternative benchmark. We also provide comparative analyses of this index across different targeting policies and across different underlying network structures.

Keywords: Network games, targeting, strategic complements, synergy.

JEL classification: D21, D29, D82

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1 Introduction

Network structure plays an important role in different types of behaviors. For instance, social communication Apps have attracted numerous users to actively participate and interact; according to Statista (2020), the numbers of active users in Facebook, WhatsApp, and WeChat are 2.7 billion, 2.0 billion, and 1.2 billion respectively. Peer effects among users have huge influence on the purchases of video games like XBox and Switch, and sales of retailing business. See the surveys by Jackson and Zenou (2015) and Jackson, Rogers, and Zenou (2017).

Given the profound impact of network effects, it is not surprising that fine-grained policies to target at selected users in the social space are increasingly adopted by firms in these markets.[1] The central premise of such *seeding*, also known as network-based targeting, is that users in different network positions have differential influences on others' behaviors. Thus, if seeding is done appropriately, the targeted users will take the lead and boost the followers' incentives, thereby resulting in a better market outcome. For example, Richardson and Domingos (2002) discuss how targeting can enhance the diffusion of brand awareness. In the charitable giving context, Jack and Recalde (2015) conducted a field experiment to show that targeting the community leaders can increase the public goods contribution. Banerjee, Chandrasekhar, Duflo, and Jackson (2019) examine how targeting improves the propagation of microfinance programs. Beaman, BenYishay, Magruder, and Mobarak (2021) investigate the effect on technology adoption in the agriculture sector.

The optimal targeting relies on the information about the social links among users, and the exact solution is often not simple to compute. Despite this, there have been many discussions on optimal targeting in the literature, both theoretically and empirically (such as Chen et al. (2009) and Kempe et al. (2005), and references in the literature review section). It is such a profitable domain that many companies have devoted full efforts to this targeting practice (such as Strawberry Socials, Yoke Network, SDA Media, to name a few), and accordingly there are plenty of websites and Apps that provide rankings of these companies.[2] On the other hand, quantitatively evaluating the values across different network structures and different targeting strategies is less explored (see the literature review).

In this paper, we introduce a novel measure to evaluate the effectiveness of any targeting policy, and provide an upper bound on the value of targeting across different strategies, and in different environments with different network structures. We build upon the elegant framework by Ballester et al. (2006) where a finite set of players with linear quadratic payoffs engage in a network game. The network structure features local peer effects and summarizes the strategic interactions among

[1] Various consultants and digital marketing advisors provide guidelines for such influencers, see, e.g., https://www.digimind.com/influencers-identification, https://shanebarker.com/blog/detailed-strategy-find-right-influencers-for-your-business/, and https://contentmarketinginstitute.com/2016/06/identify-influencers-industry/.

[2] See, for instance, "Top Influencer Marketing Companies" by https://www.businessofapps.com/marketplace/influencer-marketing/, and "38 Leading Influencer Marketing Agencies to Work With In 2021" via https://influencermarketinghub.com/top-influencer-marketing-agencies/.

players, and thus shapes the equilibrium actions. The peer effect is parameterized by a scalar called the "initial peer effect parameter." As in Zhou and Chen (2015), the network designer in our baseline model can target a subset of players such that they move first, and their decisions are observed by others.

This sequential decision making process is commonly observed in a variety of contexts. For example, in the charity giving context, a classical example is Brook Astor, an American Philanthropist, whose donations were frequently reported in the news and followed by others to copy her contribution.[3] The rationale is that people are by and large "conditional collaborators": they contribute to the public goods only when others have done so (Gächter et al. 2010).[4] In the teamwork and organizational design contexts, some individuals can lead by example, volunteer the tasks, choose efforts prior to others, and make contributions before others take actions (Hermalin 1998). In the global environmental context, Potters et al. (2007) reported that "a nation may be first to commit to participation in an international environmental agreement."

Our main result is simple to state. For any targeting problem, we can equivalently represent a positive scalar such that the efforts with such targeting/seeding has the same aggregate effort as multiplying the peer effect parameter by this scalar. We call it *relative network synergy equivalent* (RNSE).[5] The main message is this index, for any targeting policy on the network space, has a *constant* upper bound, $\sqrt{2} \approx 1.414$, which holds for any network, any intrinsic valuation profile, and any initial peer effect parameter. In other words, about 41.4% increase in the magnitude of the peer effect is the upper bound on the value of any targeting policy. Because this bound applies to the optimal targeting as well, it provides a quantitative limit on the value of network-based targeting.

Moreover, this upper bound is *tight*: there exists a scenario with RNSE which can be arbitrarily close to this upper bound. Therefore, although it is valuable to use targeting based on network information, we show that the value of such network-based targeting may be rather limited. Along the process we also show that the first-best outcome has an RNSE of 2 for every network game studied here. Therefore, the gap between the optimal targeting outcome and the first-best is universally bounded below by $1 - \sqrt{2}/2 \approx 29.3\%$. Incidentally, the pursuit of *constant* performance bound (that is independent of problem specifications) lies in the core of the computer science literature when developing approximation algorithms (Williamson and Shmoys 2011) and characterizing price of anarchy (Koutsoupias and Papadimitriou 1999).

We extend our results along various dimensions. First, we study the pricing of a monopolist selling social products. In this case, the designer's objective becomes revenue maximization rather

[3] A New York Times article states that "[w]hen she gave one donation to the New York Library, for example, three other major gifts – from Bill Blass, Dorothy and Lewis B. Cullman, and Sandra and Fred Rose – all followed, with her generosity cited as the inspiration" (March 30, 2002, p. A13).

[4] List and Lucking-Reiley (2002) showed that initial charity contribution is positively correlated with the subsequent amount. Soetevent (2005) found that for church donations, making the process open increases the total amount.

[5] This notion is similar in spirit to notions like *equivalent variation* (EV) and *compensating variation* (CV) in industrial organization, which measure the economic welfare changes associated with changes in prices.

than aggregate action maximization. Second, we examine the case when there is a cost of intervening that is increasing in the number of nodes that need to be seeded. We allow for arbitrary costs of seeding. Third, we consider an alternative benchmark – randomly seeding a fixed number of players – rather than the benchmark of everyone being a seed. Fourth, we consider the general sequential problem with multiple stages of moves; i.e., players are split into more than two groups and move sequentially according to their designated group order. Furthermore, players can be endowed with heterogeneous marginal utilities. In this case, the targeting involves the design of the entire hierarchy as well as whom to be included in each stage.

We show that $\sqrt{2}$ remains the (uniform) upper bound for all the above extensions. Though this observation does not reject the usefulness of targeting and the associated refinements, they have limited value in a variety of scenarios. As another robustness check, we consider the extension when the designer lets some seed players to revise their actions at the later stage. We find that it is not beneficial for the designer (nor any player) to allow for multiple moves and revisions. Thus, if a targeted node can choose between having the flexibility to revise or giving up this option, the targeted node will surrender the option. We also consider a variant in which targeted players can only increment their chosen actions in the later stage. We find that in this case, all targeted players will choose not to increment in the later stage. In both scenarios, being targeted and moving in the early stage complies with the players' self-interest, and the planner's arrangement will be voluntarily executed.

To apply our results, it is notable that the environments need to satisfy the following conditions: adoption decisions involve players choosing the magnitude of behavior; decisions have positive complementarities; a designer can control who moves first and who moves second; and this decision can neither be delayed nor revised by the players. As described above, even though each node can make continuous contributions in multiple opportunities, only one-time contribution for each node will be observed. The key point is that the designer or the central planner can indicate which player (players) should move first; the targeted players will follow through the planner's plan. In Section 3, we describe specific applications and practical examples that fit these conditions.

Our analysis not only suggests the limit of targeting value, but also offers a quantitative goal to invest in peer effect enhancing technology. Our proposal may be more advantageous than other tactics in certain scenarios. As aforementioned, targeting requires detailed knowledge of network structure that can be challenging to obtain. Adding more seeds randomly is handy, but it requires accurate knowledge about the total network size, and the performance can be guaranteed only in the limiting regime with very large network sizes. In contrast, our result does not rely on asymptotic arguments and holds for arbitrarily small networks.

The paper proceeds as follows. Section 2 reviews relevant literature. Section 3 presents the model with detailed analysis in Section 4. Some extensions are discussed in Section 5. Section 6 concludes. The Appendix contains some of the results and proofs omitted in the main text.

2 Literature Review

The literature on network externality started with aggregate effects (i.e., the aggregate number of players in the game is a sufficient statistics of this effect); see, e.g., Rohlfs (1974), Katz and Shapiro (1985), Farrell and Saloner (1986), and the survey by Economides (1996). Ballester et al. (2006) introduce a tractable model to incorporate the local network effects and characterize the Nash equilibrium actions through the measure weighted Katz-Bonacich Centrality. Candogan et al. (2012) and Bloch and Quérou (2013) add the monopoly firm's pricing problem to this simultaneous-move network game and investigate the benefit of discriminatory pricing.

The possibility of sequential moves among players is introduced by Zhou and Chen (2015), which shows that, (1) the benefit of sequentiality in network games with strategic complementarity and positive spillover; (2) for moderate degree of externalities, the optimal leader group selection problem, is equivalent to the weighted max-cut problem; (3) sequence refinement is always beneficial. Our paper is closely related to Zhou and Chen (2015), and we show that regardless of the network structure and the sequence of moves, the value of such targeting and sequentiality, measured by the proposed RNSE index, is limited by a constant bound.

Because in the application we allow the designer to also set network-based price discrimination, our paper is related to the literature on the value of network information in pricing, diffusion of products. Bloch (2016) offers a recent survey on targeting and pricing in social networks. Huang et al. (2019) investigate the benefit of network-based price discrimination in large random networks, and find that it decays quickly in the network size. Belhaj and Deroïan (2018, 2019) study the optimal contract design and targeting on a network with local synergies. The possibility of offering the products sequentially is studied in Aoyagi (2010). In his model, customers have interdependent values and possess some private information. Thus, sequential launch allows late customers to learn from early customers' decisions. In our paper, this social learning channel is absent, and we show that with payoff externalities sequential selling has limited value.

There are alternative types of targeting or interventions, see, among others, Bloch (2016), Demange (2017), and Galeotti, Golub, and Goyal (2020). Demange (2017) considers a setting in which players react to their neighbors' actions according to given reaction functions. The planner can intervene the network game by boosting directly players' reactions. Galeotti, Golub, and Goyal (2020) study a network game in which players' payoffs contain a private return and externalities that arise from neighbors' actions. They focus on a different type of intervention: the planner's budget is spent to increase the marginal returns of players' private payoffs. The planner's objective in Demange (2017), as in our baseline model, is to maximize the aggregate action, while in Galeotti, Golub, and Goyal (2020) the planner maximizes certain welfare objectives. Because both games in Demange (2017) and Galeotti, Golub, and Goyal (2020) involve strategic complementarities, either directly enhancing the players' reactions or indirectly raising the returns to actions will increase players' actions. Thus, targeting is necessarily beneficial, and more budgets unambiguously result in higher aggregate actions. Specifically, Galeotti, Golub, and Goyal (2020) consider both strategic complements and substitutes, and characterize the optimal targeting strategies using either the first or the last principal component of the underlying network. In contrast, targeting in our context does not impact a player's intrinsic payoff or reaction to neighbors' actions. The planner needs to leverage the positive feedback through the leader-follower interactions.

In terms of the limited value of network information, our paper is related to Akbarpour, Malladi, and Saberi (2020), who evaluate the effectiveness of seeding in diffusion of information on networks. They consider a family of diffusion models wherein players spread information stochastically, and the designer determines seeding to maximize the expected coverage (see Sadler (2020) for recent theoretical development of the diffusion game on networks). As in the Susceptible-Infected-Recovered (SIR) model, after a single exposure to someone else who has already adopted in the network, a player must adopt. They observe that in diffusion models the success of seeding utilizes the structure of clustering. Intuitively, when there are giant clusters, optimal seeding will certainly target these clusters, but random seeding will very likely hit them too. When all cluster sizes are small, even with optimal seeding the spread is rather limited. They show that adding a relatively few more seeding is better than optimally selected seeding which requires information about the network knowledge. Beaman, BenYishay, Magruder, and Mobarak (2021) point out that the above "simple contagion" condition is key, and investigate the targeting effect on technology adoption in the agriculture sector. They find that if adoptions happen only when a player is exposed to at least two neighbors who have adopted (complex contagion), targeting may significantly improve technology diffusion. In both contagion models, having more seeds weakly improves the aggregate adoptions because exposures are necessary. In a very different setting, our constant performance bound is irrespective of the clustering properties, and it applies to arbitrary networks without resorting to the limiting regimes.

Our paper speaks to some recent empirical literature on the value of network information in development economics. As discussed, Beaman et al. (2021) conducted field experiments in Malawi and evaluated the use of diffusion models on rich social networks for technology adoption. Alatas et al. (2019) tested how influential individuals' tweets can promote the vaccination coverage in Indonesia. Eckles et al. (2019) study the influence maximization problem, discuss the complexity of seeding, and evaluate the effectiveness of algorithms. See also Chin et al. (2018) for the evaluation of random seeding.

Finally, the limited value on sequentiality is also studied in the congestion games, where a group of agents determine to send their traffic flows through a physical network. In these games, agents experience congestion and share costs when their traffic flows share the same link, and the congestion level of one link is determined by the aggregate flow through it. Leme et al. (2012) introduce the sequential moves into congestion and cost-sharing games and show that the "sequential price of anarchy" can be lower than the classical price of anarchy. Swamy (2012) examines the effectiveness of two-stage sequential moves and identifies constant performance bounds for series-parallel networks, and problem-specific bounds for general networks. de Jong and Uetz (2014) focus on atomic games (i.e., individual agent's influence is not infinitesimal) and allow for general number of rounds of sequential moves. They establish some constant performance bounds using the techniques of mixed integer programming. In our paper, agents experience payoff externalities and thus the network is defined based on their inherent relationships, and they make one dimensional decisions.

3 Model

3.1 Setup

We first present a baseline model of targeting in social networks, and then introduce an index to measure the relative effectiveness of different targeting strategies.

The network game

Our baseline model setup follows closely from Ballester et al. (2006) and Zhou and Chen (2015). Consider a game played on a non-empty network among a finite set of players $\mathcal{N} := \{1, 2, \cdots, n\}$. We use the matrix $\mathbf{G} = (g_{ij})_{1 \leq i,j \leq n}$ to represent the network structure. As is standard in the literature, we assume that the network is undirected, i.e., $g_{ij} = g_{ji} \in \{0, 1\}$, and there is no self-loop, i.e., $g_{ii} = 0$.[6] Each player $i \in \mathcal{N}$, represented as a node of the network, chooses an action (effort) $x_i \in \mathbb{R}$ and obtains the following utility:

$$\pi_i(x_1, x_2, \cdots, x_n) = a_i x_i - \frac{1}{2} x_i^2 + \delta \sum_{j \in \mathcal{N}} g_{ij} x_i x_j. \tag{1}$$

The first two terms in (1) collectively capture the diminishing marginal return of the player's own action, where the parameter $a_i > 0$ measures player i's intrinsic marginal utility of own action. The last term captures the network complementarity effect among players, where the parameter $\delta > 0$ describes the strength of this peer complementarity effect. This specification of utility is popularized by Ballester et al. (2006) and has been widely adopted in many papers; see, among others, Bloch (2016) and Galeotti, Golub, and Goyal (2020). Note that the network structure $\mathbf{G}$ summarizes the strategic interactions among players, and thus shapes the equilibrium actions.

Targeting

Following Zhou and Chen (2015), we consider a network designer who can influence the sequence of moves in the underlying network game, and subsequently influence the equilibrium outcome, by targeting a subset of players $\mathcal{A} \subset \mathcal{N}$. Effectively, the targeted players in $\mathcal{A}$, called *seeds*, choose their actions earlier than those players not targeted (in $\mathcal{N} \backslash \mathcal{A}$). Formally, the targeting policy $\mathcal{A}$

[6]Our main results can be easily extended to allow for weighted directed networks.

selected by the designer determines the sequence of moves of a dynamic game specified as follows:[7][8]

- In stage 1, the players in group $\mathcal{A}$ choose actions simultaneously;
- In stage 2, the players in the rest of the network, $\mathcal{N} \backslash \mathcal{A}$, choose their actions simultaneously. The payoffs, as given (1), are realized after all actions are taken.

For this two-stage game, we apply the standard solution concept of subgame perfect Nash equilibrium (SPNE). We defer the detailed equilibrium analysis such as the condition for existence and uniqueness, and the explicit equilibrium formulae to the next section. For now, let $\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \in \mathbb{R}^n$ denote the equilibrium action profile. Denote by $x(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) := \mathbf{1}'\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \in \mathbb{R}$ the equilibrium aggregate action, where $\mathbf{1}$ is the n-vector of 1s.

Benchmark

We first analyze the benchmark case in which the designer does not target any player, i.e., $\mathcal{A} = \emptyset$[9], and thus all players choose actions simultaneously. Let $\lambda_{\max}(\mathbf{G})$ denote the largest eigenvalue of matrix G. Ballester et al. (2006) show that, when $0 < \delta < 1/\lambda_{\max}(\mathbf{G})$, the simultaneous-move network game admits a unique Nash equilibrium, which equals the Katz-Bonacich centrality measure:[10]

$$\mathbf{b}(\mathbf{G}, \delta; \mathbf{a}) := (b_1(\mathbf{G}, \delta; \mathbf{a}), \cdots, b_n(\mathbf{G}, \delta; \mathbf{a}))' = [\mathbf{I} - \delta\mathbf{G}]^{-1}\mathbf{a}. \tag{2}$$

Note that $b(\mathbf{G}, \delta; \mathbf{a}) = \mathbf{1}'\mathbf{b}(\mathbf{G}, \delta; \mathbf{a})$ is *aggregate equilibrium effort*. This will be the network designer's objective in our base model. Maximizing aggregate action is a common goal for team production, charity giving, public good contribution, and environmental policy engagement. It is easy to see that, due to the strategic complementarity, each individual action $b_i(\mathbf{G}, \delta; \mathbf{a})$ and the aggregate effort $b(\mathbf{G}, \delta; \mathbf{a})$ strictly increase with δ.[11] We exploit this monotonicity observation to define the following index.

[7] We assume that the underlying network structure is common knowledge. Also, the set of seeds, once chosen by the designer, is commonly known among all players.

[8] The targeting in our setting is closely linked to Stackelberg games and the value of commitment (see Bagwell (1995)). See Section 4.5 for further discussions.

[9] Equivalently, the designer targets the whole network, i.e., $\mathcal{A} = \mathcal{N}$.

[10] Players' best responses are determined by the first-order conditions of (1): $x_i = a_i + \delta \sum_{j \in \mathcal{N}} g_{ij} x_j, \;\; i \in \mathcal{N}$, which leads to the equilibrium action profile $\mathbf{x} = [\mathbf{I} - \delta\mathbf{G}]^{-1}\mathbf{a}$.

[11] More precisely, the *inverse Leontief matrix* associated with network $\mathbf{G}$ and parameter δ is

$$\mathbf{M}(\mathbf{G}, \delta) := [\mathbf{I} - \delta\mathbf{G}]^{-1}. \tag{3}$$

Using a path-counting interpretation in Ballester et al. (2006), we observe that each entry m_{ij} of $\mathbf{M}$ represents the discounted number of walks from i to j in network $\mathbf{G}$ with decay factor δ. When $0 < \delta < 1/\lambda_{\max}(\mathbf{G})$,

$$[\mathbf{I} - \delta\mathbf{G}]^{-1} = \mathbf{I} + \sum_{k=1}^{\infty} \delta^k \mathbf{G}^k. \tag{4}$$

From the above expression, it is obvious to see that b_i increases with δ.

Definition 1. *For each $\mathcal{A} \subseteq \mathcal{N}$, let $\gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ be the unique positive scalar $\gamma \in \mathbb{R}_+$ such that*

$$x(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) = b(\mathbf{G}, \gamma\delta; \mathbf{a}). \tag{5}$$

We call γ the ***relative network synergy equivalent (RNSE)*** *of targeting $\mathcal{A}$.*

The basic idea of the definition of RNSE is straightforward. Fixing any network, a targeting/seeding policy may induce a different aggregate action than the benchmark outcome $b(\mathbf{G}, \delta; \mathbf{a})$. The index γ calibrates the multiplicative scalar of the network synergy parameter δ in the simultaneous-move benchmark game with the same level of aggregate action. In addition, $b(\mathbf{G}, \gamma\delta; \mathbf{a})$ strictly increases with γ, and approaches infinity when $\gamma \to \frac{1}{\delta\lambda_{\max}(G)}$, implying that the γ defined in (5) exists, and is unique.

Zhou and Chen (2015) consider the following optimal targeting problem:

$$\max_{\mathcal{A} \subseteq \mathcal{N}} x(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}). \tag{6}$$

In other words, the designer's objective is to choose the set of seeds $\mathcal{A}$ to maximize the aggregate action. Using the definition of γ, we have the following observation.

Lemma 1. *The designer's problem in (6) is equivalent to*

$$\max_{\mathcal{A} \subseteq \mathcal{N}} \gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}). \tag{7}$$

Instead of solving for the optimal seeding strategy on a fixed network (which is analyzed in Zhou and Chen (2015)), we are primarily interested in providing a *tight upper bound* for γ, which shows the limit of targeting in networks.

Notation. Before we proceed, we introduce some notation that will be used throughout the paper. For a matrix $\mathbf{T}$, the transpose is denoted $\mathbf{T}'$. The zero matrix/vector (of suitable dimensions) is denoted $\mathbf{0}$. If $\mathbf{T}$ is a square matrix, then $\mathbf{T}^D$ is a matrix with diagonal entries $T^D_{ii} = t_{ii}, i = 1, \cdots N$, and off-diagonal entries $T^D_{ij} = 0$, $\forall i \neq j$. Unless otherwise indicated, the vector $\mathbf{x} = (x_1, \cdots, x_N)'$ is a column vector. For any subset S of $\mathcal{N}$, $\mathbf{x}_S$ (in bold) denotes the vector $(x_i)_{i \in S}$; that is, it is a sub-vector wherein the sequence of selected components follows their original sequence in the vector $\mathbf{x}$. The (non-bold) term $x_S = \sum_{i \in S} x_i$ is the sum of these selected components. We let $\langle \mathbf{x}, \mathbf{y} \rangle$ denote the inner product of two column vectors $\mathbf{x}, \mathbf{y}$. We say that two matrices $\mathbf{T}^1$ and $\mathbf{T}^2$ satisfy $\mathbf{T}^1 \succeq \mathbf{T}^2$ if and only if $T^1_{ij} \geq T^2_{ij}$ for all i, j, and $\mathbf{T}^1 \succ \mathbf{T}^2$ if and only if $\mathbf{T}^1 \succeq \mathbf{T}^2$ and $T^1_{ij} > T^2_{ij}$ for some i, j. In other words, these dominance relationships apply to the component-wise comparisons. Given a real-valued function f, and an integer k, we write $f(\delta) = \mathcal{O}(\delta^k)$ if $\limsup_{\delta \to 0} \left| \frac{f(\delta)}{\delta^k} \right| < \infty$. When $\mathbf{a} = \mathbf{1}_n$, we write $b(\mathbf{G}, \delta; \mathbf{a})$ as $b(\mathbf{G}, \delta)$, $x(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ as $x(\mathbf{G}, \delta, \mathcal{A})$, and $\gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ as $\gamma(\mathbf{G}, \delta, \mathcal{A})$.

3.2 Preliminary results

Below we discuss several properties of the index γ.

Remark 1.

(i) *Scaling the marginal utility parameters* $\mathbf{a}$ *to* $t\mathbf{a}$ *(where* $t > 0$*) does not change the value of* γ*. The fact that the index* γ *is a scalar enables us to make meaningful comparisons across targeting policies within a fixed network, as well as across different networks for a fixed targeting policy.*

(ii) *Proposition 1 in Zhou and Chen (2015) shows that targeting always increases the equilibrium action profile, i.e.,* $x(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \geq b(\mathbf{G}, \delta; \mathbf{a})$*. As a consequence, the index has a lower bound:* $\gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \geq 1$*.*

(iii) *Let the* first-best *effort profile be* $\mathbf{x}^{FB}(\mathbf{G}, \delta; \mathbf{a})$*, which maximizes the aggregate payoff*

$$W(\mathbf{x}) := \sum_{i=1}^{n} u_i = \mathbf{a}'\mathbf{x} - \frac{1}{2}\mathbf{x}'\mathbf{x} + \delta\mathbf{x}'\mathbf{G}\mathbf{x}.$$

The first-best effort profile is $\mathbf{x}^{FB}(\mathbf{G}, \delta; \mathbf{a}) = [\mathbf{I} - 2\delta\mathbf{G}]^{-1}\mathbf{a} = \mathbf{b}(\mathbf{G}, 2\delta; \mathbf{a})$*.*[12] *In the same spirit of Definition 1, we can define the RNSE index of the first-best outcome as* $\gamma^{FB} = 2$*. As players in the benchmark case do not fully internalize the effects of their actions on other players, under-provision of efforts, compared with the first-best outcome, is expected.*

(iv) *The equilibrium outcome* $\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ *depends on* $\mathbf{G}, \delta, \mathcal{A}$*, and* $\mathbf{a}$*, so is* γ*. Obtaining a closed-form expression of* γ *is generally hard. Nevertheless, our goal is to characterize a tight upper bound of* γ *for a wide range of network games.*

(v) *Our RNSE index* γ *is micro-founded using the calibration exercise of the Katz-Bonacich centrality measures (Ballester et al. 2006). When the set* $\mathcal{A}$ *is a singleton, the Katz-Bonacich centrality* b_i*, the inter-centrality index* $c_i = \frac{b_i^2}{m_{ii}}$*, the key leader index* $L_i = \frac{b_i^2}{m_{ii}}\frac{m_{ii}-1}{2-m_{ii}}$*, and our RNSE index* $\gamma(\{i\})$ *are closely linked together.*[13] *For any regular network, we have* $b_i = b_k$ *for any* i, k*. Furthermore, we can show the following simple relationship:*

$$\gamma(\{i\}) > \gamma(\{k\}) \iff L_i > L_k \iff m_{ii} > m_{kk} \iff c_i < c_k.$$

In other words, for any regular network in which b_i *is constant, both our RNSE index* $\gamma(\{i\})$ *and the key leader index* L_i *are correlated perfectly and positively with the self-loop* m_{ii}*, while*

[12] Here $\mathbf{x}^{FB}$ solves the first-order conditions: $\mathbf{a} - \mathbf{x}^{FB} + 2\delta\mathbf{G}\mathbf{x}^{FB} = \mathbf{0}$, implying $\mathbf{x}^{FB} = [\mathbf{I} - 2\delta\mathbf{G}]^{-1}\mathbf{a} = \mathbf{b}(\mathbf{G}, 2\delta; \mathbf{a})$, which is well-defined only when $0 < \delta < 1/(2\lambda_{\max}(G))$ (this condition also guarantees the strict concavity of W).

[13] Ballester et al. (2006) characterize c_i as the solution to the key player problem in criminal networks. Zhou and Chen (2015) characterize L_i as the solution to the key leader problem. We assume, for simplicity, $a_i = 1$ for all i.

the intercentrality index c_i is correlated perfectly and negatively with m_{ii}. Since it is easy to construct a regular network[14] *with heterogeneous m_{ii}'s, we can have instances in which the correlation between $\gamma(\{i\})$ and b_i or the correlation between $\gamma(\{i\})$ and c_i are easily described.*

Remark 1(iii) suggests that we can also define an index when comparing the gap between the sequential launch outcome with the first-best scenario, and it has a simple expression: $1 - \gamma/2 \in [0, 0.5]$. Opposite to γ, the higher the term $1 - \gamma/2$ is, the worse performance we obtain. Notably, in the class of network games we consider, the equilibrium analysis requires a certain stability condition on δ; i.e., $\delta > 0$ cannot be too large (see Definition 2 and Corollary 1).

3.3 Discussions

The above model has some unique ingredients and deserve some further discussions. As aforementioned, it requires (1) adoption decisions involve players choosing the magnitude of behavior; (2) decisions have positive complementarities; (3) a designer can control who moves first and who moves second; and (4) this decision can neither be delayed nor revised by the players. Below, we highlight two contexts for which these conditions are appropriate: crowdfunding and charity giving.

Crowdfunding. Crowdfunding platforms host projects initiated by creators, and a project will be realized only when the total funds it attracts from interested buyers reach a specified target (Hu et al. (2015)). Interested buyers can choose to commit to an arbitrary amount (from 30 pounds to over 150 pounds for a book in 2006, see Hu et al. (2015)); higher amounts result in quicker accumulation of the total fund, lead to better chance for project success, and may bring higher private benefits to the buyers (in the form of up-scale product or additional gifts). Crowdfunding platforms such as Kickstarter typically publish the cumulative fund over time; thus, the decisions of the earlier contributors (buyers) are observable to later buyers.

As described above, crowdfunding mechanisms have a built-in positive complementarity (projects succeed only when the pledged target is met). There can also be inherent externalities from buyers, too. For example, "[o]n the demand side, the product may exhibit positive externality" (Hu et al. (2015)). Because contributions are solicited before the product is brought to market, buyers must make investment decisions when facing information asymmetry (Chakraborty and Swinney (2021)). As some buyers are more informed than others, observing the decisions by others can generate information externality. Collectively, we feel that crowdfunding satisfies (1)–(4). Local network effects again can arise when players value some decisions more than others.

Charity giving. Jack and Recalde (2015) study the public good provision in 52 communities in Bolivia. The choice of each player is how much money to contribute to provide environmental education books for the local school, and "even small contributions could ensure positive levels of provision." Thus, it satisfies (1): the contributions are not binary, but rather a magnitude of behavior. Environmental education books are non-excludable and generate social spillovers because

[14] See, for instance, Figure 1 in Zhou and Chen (2015).

"all community members can access the education material available in the local school." Thus, "donations generate a positive externality that is non-rival." This satisfies (2): decisions have positive complementarities.

Moreover, in these communities "an authority makes an initial public voluntary contribution – and other groupmembers make private voluntary contributions after observing the authority's choice." Because participants "are given the opportunity to make voluntary contributions to acquire environmental education books for the local school," once a contribution is made, each player cannot revoke or revise the amount. This simultaneously satisfies (3) and (4): the designer can control who moves first and who moves second, and this decision can neither be delayed nor revised by the players.

Jack and Recalde (2015) also make empirical observations that are consistent with the theoretical predictions in this literature: targeting the community leaders can increase the public goods contribution (compared with the simultaneous benchmark), and "follower contributions are less strongly influenced by the contributions of randomly selected leaders." This implies that network effects are local rather than global, because players have differential influences on or are differentially affected by others.

4 Analysis

4.1 An illustrating example: a dyad network

Before stating our main results in the general network, we first consider the simplest case of a dyad network with two players, where $\mathbf{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Example 1. *In a dyad network, the equilibrium actions in the simultaneous-move game are*

$$x_1(\delta) = b_1(\delta) = \frac{a_1 + \delta a_2}{1-\delta^2},\ x_2(\delta) = b_2(\delta) = \frac{a_2 + \delta a_1}{1-\delta^2}.$$

Suppose $\mathcal{A} := \{1\}$; i.e., player 1 moves first, then 2 follows. The equilibrium actions in the sequential game are[15]

$$x_1^S(\delta) = \frac{a_1 + \delta a_2}{1-2\delta^2}, \quad x_2^S(\delta) = a_2 + \delta x_1^S = \frac{\delta a_1 + (1-\delta^2)a_2}{1-2\delta^2}. \tag{8}$$

Assume $a_1 = a_2 = 1$. Observe that

(a) For the leader (player 1)'s action, we have

$$\frac{1+\delta}{1-2\delta^2} = x_1^S(\delta) < b_1(\sqrt{2}\delta) = \frac{1+\sqrt{2}\delta}{(1-(\sqrt{2}\delta)^2)}.$$

[15] Anticipating the player 2's best response, $a_2 + \delta x_1$, the player 1, as the first mover, chooses x_1^S to maximize $a_1 x_1 - \frac{1}{2}x_1^2 + \delta x_1(a_2 + \delta x_1)$, which leads to x_1^S as given in (8).

(b) *Similarly, for the follower (player 2), we have*

$$x_2^S(\delta) = 1 + \delta x_1^S(\delta) < 1 + \delta b_1(\sqrt{2}\delta) < 1 + \sqrt{2}\delta b_1(\sqrt{2}\delta) = b_2(\sqrt{2}\delta),$$

where the first equality is by the best response of player 2, who moves after the player 1 in the sequential game with synergy parameter δ; the first inequality is by previous step (a); the second inequality is by $\sqrt{2} > 1$; and the last equality is by best-response of player 2 with synergy parameter $\sqrt{2}\delta$ in the static NE.

(c) *Combining items (a) and (b) yields*

$$\mathbf{x}^S(\delta) \prec \mathbf{b}(\sqrt{2}\delta),$$

which implies that

$$\gamma(\mathbf{G}, \delta; \mathcal{A}) < \sqrt{2}.$$

(d) *By Definition 1, γ is the scalar that satisfies $x_1^S(\delta) + x_2^S(\delta) = b_1(\gamma\delta) + b_2(\gamma\delta)$, which reduces to the following (recall that we have assumed $a_1 = a_2 = 1$)*

$$\frac{2(1+\delta) - \delta^2}{1 - 2\delta^2} = 2\frac{1 + \gamma\delta}{1 - \gamma^2\delta^2}.$$

In other words,

$$\gamma(\mathbf{G}, \delta; \mathcal{A}) := \frac{2 + 3\delta}{2 + 2\delta - \delta^2}. \tag{9}$$

The RHS of (9) increases in δ, and it equals 1 when $\delta = 0$, and converges to $\sqrt{2}$ as $\delta \to \sqrt{1/2}$ (see Figure 1).[16]

In this example, an upper bound of γ is $\sqrt{2}$. This example also admits a lower bound of 1, and it brings in the negative news. Namely, there exists a scenario in which the improvement by the optimal targeting is minimal, and the gap from the first-best outcome attains the maximum value 50%. This worst case occurs when the initial peer effect parameter δ is very small, i.e., the synergy is negligible.

Our main analysis mostly explores the benefit of targeting on promoting aggregate effort. Sometimes a well-intended designer may have alternative objectives in mind when constructing the optimal targeting; for instance, she may want to reduce inequality in the outcome (or welfare) distribution. As a detour before our main result, we illustrate the implications of targeting on alternative objectives such as inequality/distribution in the following example.

Example 2. *Consider the Dyad network as in Example 1 and let $\mathcal{A} = \{1\}$. We discuss two cases.*

[16] Note that, to satisfy permissibility (see Definition 2), the range of δ is $[0, 1/\sqrt{2})$.

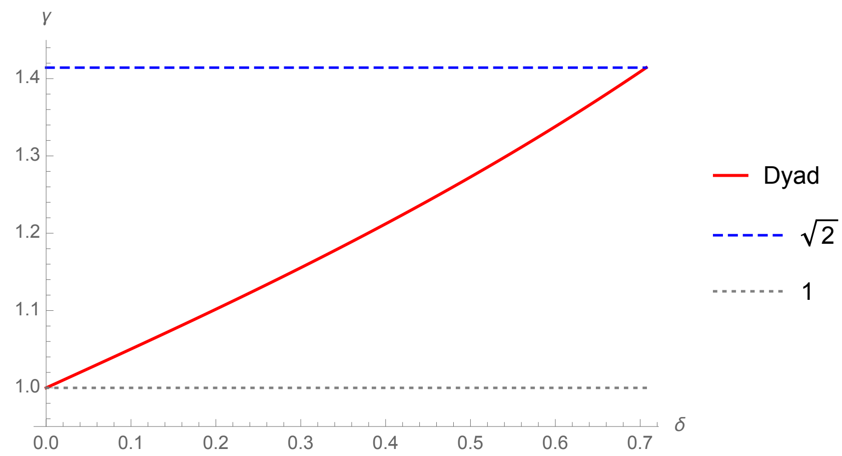

Figure 1: Dyad network. The red solid curve is the index $\gamma(\cdot)$ given in (9). The x-axis is δ. Two horizontal lines are lower bound 1 (in dotted grey), and upper bound $\sqrt{2}$ (in dashed blue).

(i) Suppose $\frac{a_1}{a_2} = 1$. In the benchmark case with no targeting, the equilibrium effort is symmetric – there is no inequality. However, with $\mathcal{A} = \{1\}$ targeted, player 1's equilibrium action is strictly higher than that of the follower (player 2); i.e., $x_1^S > x_2^S$.

(ii) Suppose $\frac{a_1}{a_2} = \frac{1-\delta-\delta^2}{1-\delta}$, with $0 < \delta < 0.618$ so that $1 - \delta > 1 - \delta - \delta^2 > 0$. In the benchmark case with no targeting, player 1 exerts a strictly lower effort: $b_1 < b_2$ as $a_1 < a_2$. On the other hand, with $\mathcal{A} = \{1\}$ targeted, both players exert the same amount of effort in the unique SPNE outcome.

Observe that targeting can either increase inequality as in case (i) or reduce it as in case (ii).

4.2 Main results: A universal and tight upper bound for γ

First, we start with a result in Zhou and Chen (2015):

Lemma 2. *For any $\mathcal{A} \subseteq \mathcal{N}$ that is neither $\emptyset$ nor $\mathcal{N}$,*

$$\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \succ \mathbf{b}(\mathbf{G}, \delta; \mathbf{a}).$$

In other words, targeting is always valuable to the designer.

When $\mathcal{A} = \emptyset$ or $\mathcal{N}$, obviously $\mathbf{x}(G, \delta, \mathcal{A}; \mathbf{a}) = \mathbf{b}(G, \delta; \mathbf{a})$. Targeting has no additional value when either none or all agents are selected as targets. Henceforth we focus on the case $\mathcal{A} \neq \emptyset$ or $\mathcal{N}$. The intuition behind Lemma 2, as shown in Zhou and Chen (2015), is quite simple. Note that a player's payoff is supermodular in her own action and her neighbors' actions, and exhibits positive spillovers. Due to strategic complementarity, the followers' responses in the second stage monotonically increase with the actions taken by the seeds (or first-movers, leaders) in the first stage. Due to positive spillovers, the seeds, anticipating these strategic responses from followers, have stronger incentive to exert higher actions than what they would do in the simultaneous-move case.[17] As a result, targeting is always valuable: for any network $\mathbf{G}$ and any $\mathcal{A} \subseteq \mathcal{N}$ that is neither $\emptyset$ nor $\mathcal{N}$,

$$\gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) > 1.$$

Definition 2. *A two-stage network game* $(\mathbf{G}, \delta; \mathcal{A})$ *is* permissible *if it has a (bounded) SPNE; i.e., the equilibrium actions do not go to* $+\infty$.

Next, we present a uniform upper bound of γ for the class of permissible two-stage network games we study.

Theorem 1. *For any permissible two-stage network game* $(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$*,*

$$\gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) < \sqrt{2}.$$

Moreover, the next theorem shows that this upper bound is tight.

Theorem 2. *For any* $\epsilon > 0$*, there exists a permissible two-stage network game* $(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ *with*

$$\gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) > \sqrt{2} - \epsilon.$$

These findings directly imply the following.

Corollary 1. *The two-stage network game* $(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ *is permissible if*

$$\delta < \left(\frac{1}{\sqrt{2}\lambda_{max}(\mathbf{G})} \right).$$

Corollary 1 provides a simple-to-check sufficient condition for permissibility in Definition 2, which does not rely on $\mathcal{A}$. However, the stated condition is usually not necessary for specific $\mathcal{A}$ and $\mathbf{G}$.

Corollary 2. *Regardless of the selection of* $\mathcal{A}$*, the targeted sequential launch cannot achieve first-best.*

Proof: The first-best outcome can be viewed as a policy with $\gamma^{FB} = 2$ (see Remark 1 (iii)). The result follows directly as $\gamma < \sqrt{2} < 2$. □

Finally, we can provide an upper bound for the value of the optimal targeted sequential launch problem.

[17] See Echenique (2004) for a related discussion of strategic complementarities in extensive-form games.

Corollary 3. *For all* $\mathbf{G}$ *and* $\delta \in [0, 1/(2\lambda_{\max}(\mathbf{G})))$,

$$\max_{\mathcal{A}\subseteq\mathcal{N}} x(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) < b(\mathbf{G}, \sqrt{2}\delta; \mathbf{a}) < x^{FB}(\mathbf{G}, \delta; \mathbf{a}).$$

Proof: Since $\gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) < \sqrt{2} < 2$ for any $\mathcal{A} \subseteq \mathcal{N}$, the result just follows. □

We discuss some economic implications of Theorems 1 and 2.

(i) The identified upper bound of γ is *"universal"* as it applies to any network $\mathbf{G}$, any targeting policy $\mathcal{A}$, and any profile of marginal utility $\mathbf{a}$. Indeed, this bound does not require players' marginal utility from action to be homogenous. And it holds even without any knowledge of the underlying network.

(ii) Consider a designer who faces the choice between two options: (A) Synergy improvement technology that increases δ by $\sqrt{2} - 1 \approx 41.4\%$; (B) Sequential launch using network-based targeting. Our analysis shows that the benefit of option (A) dominates (B) as it induces higher total action. However, whether option (A) is indeed better than (B) depends on other economic factors. One obvious concern, which we do not consider in the baseline model, is the cost side. For option (A), the cost is the investment in synergy-improving technology. For option (B), typical cost includes implementation of targeting intervention, the computational cost of finding the optimal targeting policy, and the resources spent to collect data about the underlying network. We explicitly model the cost of seeding in an extension in Section 5.2.

4.3 Main idea behind the proof of Theorem 1

The key result is Theorem 1. The main idea of its proof is as follows.

Fix any network $\mathbf{G}$, $\gamma > 0$ that satisfies Definition 2 , and $\mathbf{a} \in \mathbb{R}^n_{++}$. Consider the two-stage sequential game with players in $\mathcal{A} \subseteq \mathcal{N}$ are targeted and move first, and players in $\mathcal{B} = \mathcal{N}\backslash\mathcal{A}$ follow after observing their actions.

The equilibrium action profile in the simultaneous-move game with a scaled parameter $\gamma\delta > 0$ is

$$\mathbf{b}(\gamma\delta) = \mathbf{M}(\gamma\delta)\mathbf{a} = [\mathbf{I} - \gamma\delta\mathbf{G}]^{-1}\mathbf{a},$$

which is positively linear in $\mathbf{a}$ and strictly increasing in γ. The SPNE action profile in the associated two-stage game with the unscaled synergy parameter $\delta > 0$ is

$$\mathbf{x}^S(\delta) = \mathbf{Z}(\delta)\mathbf{a},$$

which is also positively linear in $\mathbf{a}$. Here the matrix $\mathbf{Z}$ is determined in Zhou and Chen (2015), and it depends on $\mathbf{G}, \mathcal{A}$, and δ.

By definition, the RNSE index $\gamma(\mathbf{G}, \mathcal{A}, \delta; \mathbf{a})$ is a positive multiplicative scaling of the synergy parameter δ, where the aggregate equilibrium action in the simultaneous-move game with synergy $\gamma\delta$ and in two-stage sequential game with synergy δ are equal, i.e.,

$$\mathbf{1}'\mathbf{b}(\gamma\delta) = \mathbf{1}'\mathbf{M}(\gamma\delta)\mathbf{a} = \mathbf{1}'\mathbf{Z}(\delta)\mathbf{a} = \mathbf{1}'\mathbf{x}^S(\delta).$$

Notice that if we can show that for all permissible $(\mathbf{G}, \mathcal{A}, \delta)$,

$$(\star) \qquad \mathbf{M}(\sqrt{2}\delta) \succ \mathbf{Z}(\delta),$$

then $\sqrt{2} > \gamma(\mathbf{G}, \mathcal{A}, \delta; \mathbf{a})$ holds for all permissible $(\mathbf{G}, \mathcal{A}, \delta)$ and all $\mathbf{a}$ (see Appendix B.1).

To prove statement $(\star)$, we divide $\mathbf{Z}(\delta)$ and $\mathbf{M}(\sqrt{2}\delta)$ (as well as $\mathbf{G}$) into four sub-blocks by the leader group $\mathcal{A}$ and the follower group $\mathcal{B}$.[18] The former is the SPNE characterized by Zhou and Chen (2015). Under block-by-block comparison of $\mathbf{Z}(\delta)$ and $\mathbf{M}(\sqrt{2}\delta)$, after eliminating common terms, a sufficient condition for $(\star)$ is

$$\sqrt{2}\delta\mathbf{T}(\sqrt{2}\delta) \succ \delta(\mathbf{T}(\delta) + \mathbf{T}(\delta)^D) \tag{10}$$

where

$$\mathbf{T}(\delta) = \mathbf{G}_{AA} + \delta\mathbf{G}_{AB}[\mathbf{I} - \delta\mathbf{G}_{BB}]^{-1}\mathbf{G}_{BA}.$$

(Recall that Q^D denotes the diagonal component of a matrix Q.) The first (constant) term of $\mathbf{T}$ satisfies $\sqrt{2}\delta\mathbf{G}_{AA} \succsim \delta(\mathbf{G}_{AA} + \mathbf{G}_{AA}^D)$, because all diagonal terms of the network matrix $\mathbf{G}_{AA}$ are 0. For the second term of $\mathbf{T}$ involving the parameter δ, we observe that all matrices are non-negative and $[\mathbf{I}-\delta\mathbf{G}_{BB}]^{-1}$ is increasing in δ. Substituting $\mathbf{T}(\delta)$ and $\mathbf{T}(\sqrt{2}\delta)$ into (10), it is straightforward to verify that the proposed dominance relation hold. The details of the proof is relegated to appendix A.1.

4.4 The case with small synergy

Define

$$\mathsf{CUT}(\mathcal{A}) := \sum_{i\in\mathcal{A}} \sum_{j\in\mathcal{N}\setminus\mathcal{A}} g_{ij},$$

where $\mathsf{CUT}(\mathcal{A})$ counts the number of links across agents in $\mathcal{A}$ and its complement $\mathcal{N}\setminus\mathcal{A}$.

[18] That is, we can express the matrices as $\mathbf{G} = \begin{pmatrix}\mathbf{G}_{AA} & \mathbf{G}_{AB}\\ \mathbf{G}_{BA} & \mathbf{G}_{BB}\end{pmatrix}$, $\mathbf{Z}(\delta) = \begin{pmatrix}\mathbf{Z}_{AA} & \mathbf{Z}_{AB}\\ \mathbf{Z}_{BA} & \mathbf{Z}_{BB}\end{pmatrix}$, and $\mathbf{M}(\sqrt{2}\delta) = \begin{pmatrix}\mathbf{M}_{AA} & \mathbf{M}_{AB}\\ \mathbf{M}_{BA} & \mathbf{M}_{BB}\end{pmatrix}$.

Lemma 3. *Suppose $a_i = a$ for all $i \in \mathcal{N}$, the following holds:*[19]

$$\gamma(\mathbf{G}, \delta, \mathcal{A}) = 1 + \left(\frac{\mathsf{CUT}(\mathcal{A})}{\mathbf{1}'\mathbf{G}\mathbf{1}}\right)\delta + \mathcal{O}(\delta^2).$$

There exists a cutoff $\bar{\delta} > 0$ such that, for any $0 < \delta < \bar{\delta}$, $\gamma(\mathbf{G}, \delta; \mathcal{A})$ strictly increases with δ.

When δ is small, $\gamma(\mathbf{G}, \delta, \mathcal{A})$ has a simple approximation form, which is linear in δ with the coefficient equaling the ratio of $\mathsf{CUT}(\mathcal{A})$ and $\mathbf{1}'\mathbf{G}\mathbf{1}$. Note $\mathbf{1}'\mathbf{G}\mathbf{1}$ equals two times the total number of links in $\mathbf{G}$. This simple approximation formula gives several economic insights about the effects of δ and the underlying network structure on the index. If $\delta = 0$, each player has a dominant strategy, and thus targeting has no impact at all. When $\delta > 0$, the Stackelberg incentive of leaders/seeds drives the positive value of targeting. The fact that the index $\gamma(\mathbf{G}, \delta; \mathcal{A})$ strictly increases with δ suggests that network targeting is more useful in some settings than others.[20]

Fixing a targeting policy $\mathcal{A}$, as we add more links to the network $\mathbf{G}$, both the number of cross links, $\mathsf{CUT}(\mathcal{A})$, and the total number of links, which is one half of $\mathbf{1}'\mathbf{G}\mathbf{1}$, increase. The ratio above, however, can potentially move in either direction. In the observation below we summarize two cases in which the change of the ratio, hence the index γ for small enough δ, has a predictable direction.

Observation 1.

(i) Suppose all the new links are built within the non-targeted nodes in $\mathcal{N}\backslash\mathcal{A}$, so $\mathsf{CUT}(\mathcal{A})$ remains the same, while $\mathbf{1}'\mathbf{G}\mathbf{1}$ increases. In this case, the γ index clearly goes down. Similarly, γ index goes down if all the new links are formed within the targeted nodes in $\mathcal{A}$.

(ii) Suppose all the additional links are built across $\mathcal{A}$ and $\mathcal{N}\backslash\mathcal{A}$. Then both $\mathsf{CUT}(\mathcal{A})$ and the total numbers of links go up. But it is easy to show that the ratio always increases, implying that γ index goes up.[21]

We illustrate these two cases in the following Example.

Example 3. *Suppose there are three nodes in the network, the seeding set $\mathcal{A} = \{1\}$, and agents have homogeneous marginal utility normalized to 1 (i.e., $a_i = 1$ for all i). Consider respectively the complete network (Figure 2), star network (Figure 3), and an extended Dyad network with a third isolated node (Figure 4).*[22]

[19] The extension of Lemma 3 with heterogeneous a_i's is straightforward.

[20] We are grateful to Evan Sadler for this observation.

[21] Suppose k additional links are formed between $\mathcal{A}$ and $\mathcal{N}\backslash\mathcal{A}$, then $\mathsf{CUT}(\mathcal{A})$ increases by k and $\mathbf{1}'\mathbf{G}\mathbf{1}$ increases by $2k$. The new ratio $\frac{\mathsf{CUT}(\mathcal{A})+k}{\mathbf{1}'\mathbf{G}\mathbf{1}+2k} \geq \frac{\mathsf{CUT}(\mathcal{A})}{\mathbf{1}'\mathbf{G}\mathbf{1}}$ as the original ratio is always less than 1/2.

[22] The associated network structures are

$$\mathbf{G}^c = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \mathbf{G}^* = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{G}^{Dyad} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

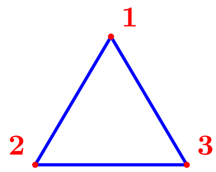

Figure 2: $\mathbf{G}^c$

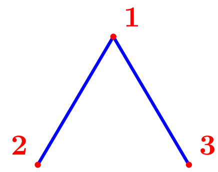

Figure 3: $\mathbf{G}^*$

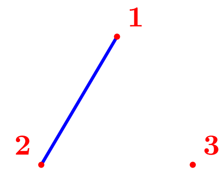

Figure 4: $\mathbf{G}^{Dyad}$

Appendix A.4 provides computations for the NE in the simultaneous-move game and the SPNE in the sequential-move game for each of the networks. Using its definition, we can obtain the exact $\gamma(\mathbf{G}, \delta; \mathcal{A})$ *for each case. Figure 3 below compares the curves of index* γ *as functions of* δ *among these three networks. We list a few observations:*

(i) For small δ*, the* γ *functions for the star and Dyad networks coincide and are valued higher than the curve for the complete network, confirming the intuition of Lemma 3.*

(ii) For larger δ*, the ranges of permissible* δ *for the three networks can differ. The* γ *index (as a function of* δ*) of the star network dominates those of the complete network and the extended Dyad network, for all permissible* δ*. Moreover, between the complete and extended Dyad networks, neither of the associated* γ *indices dominates the other for all permissible* δ*.*

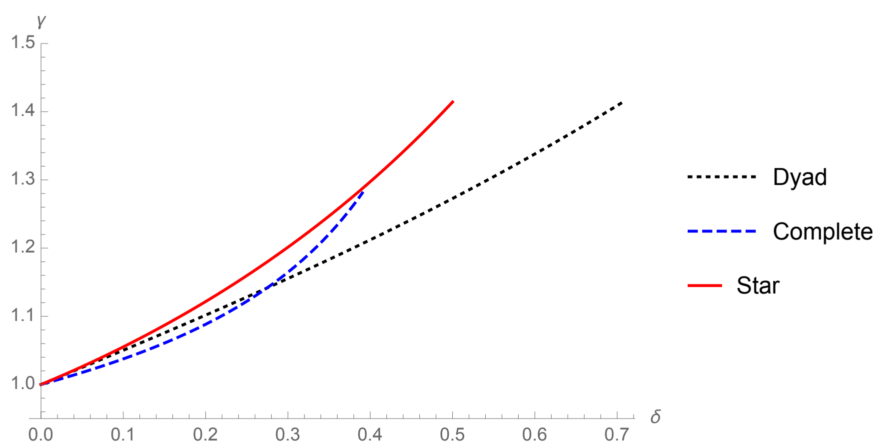

Figure 5: Plot of γ as functions of δ for the complete, star, and extended Dyad networks, where $n = 3$ and $\mathcal{A} = \{1\}$. The dotted black curve is the γ function in the extended Dyad network. The solid red curve is the γ function in the star network. The dashed curve in blue is the γ function in the complete network.

Finally, for small δ*, we apply Lemma 3 to compute the linear approximations of* $\gamma(\cdot)$ *in Table 3 below. The comparison between the three linear approximations exactly matches observation (i) in the figure above.*

network	CUT	# of links	approx. $\gamma(\cdot)$
Complete $\mathbf{G}^c$	2	6	$1+\frac{1}{3}\delta$
Star $\mathbf{G}^*$	2	4	$1+\frac{1}{2}\delta$
Dyad $\mathbf{G}^{Dyad}$	1	2	$1+\frac{1}{2}\delta$

Remark 2. *Zhou and Chen (2015) show that, when the synergy parameter δ is moderate, Problem (6) is equivalent to the following MAX-CUT problem:*

$$\max_{\mathcal{A}\subset\mathcal{N}} \mathsf{CUT}(\mathcal{A})$$

For general networks, solving the above MAX-CUT problem is computationally hard as it is a well-known NP-hard problem. That is, the complexity of Program (6) quickly explodes when we increase the network size n. Thus, for practical purposes, we have to resort to heuristics to find a targeting strategy $\mathcal{A}$ that is reasonably close to the optimal one. In contrast, the upper bound of γ we find in Theorem 1 does not require knowing the optimal $\mathcal{A}^*$. This is particularly desirable because it bypasses the challenging computational problem and does not require the detailed knowledge of the network structure.

4.5 Revisions and commitment

In this section, we relax the assumption that each node is only allowed to move once. The designer lets some seed players to revise their actions at the later stage, while payoffs are still determined by the final action chosen.[23]

Suppose that the designer allows some seed players in $\mathcal{A}$ to move again in stage 2. Formally, the designer chooses nodes $\mathcal{A}\subseteq\mathcal{N}$ to move in stage 1 and $\mathcal{B}\subseteq\mathcal{N}$ to move in stage 2, where $\mathcal{A}\cup\mathcal{B}=\mathcal{N}$. If $\mathcal{A}\cap\mathcal{B}=\emptyset$, then we are back to our main sequential model. Nevertheless, if some seeds are also in the follower set $\mathcal{B}$, then the game allows for revision of actions by these players. That is, every player i in $\mathcal{A}\cap\mathcal{B}$ can act twice: He can first choose an action $x_{i,1}$ at stage 1, which is observed by all players at the beginning of stage 2, and then revise this action to some other $x_{i,2}$ at stage 2. We denote by $x_{i,t}$ player i's action if he is called upon to move at stage $t=1,2$. Assume payoffs are determined only by the last action taken by each player. That is, let $\mathbf{x}=((x_{j,1})_{j\in\mathcal{A}\setminus\mathcal{B}},(x_{j,2})_{j\in\mathcal{B}})$ be the profile of last actions chosen by players, and the payoff function $u_i(\cdot)$ depends only on this action profile $\mathbf{x}$. Assume all these aspects of the game are common knowledge to all players.

We start with a simple example to illustrate the main idea.

Example 4. *Consider again the Dyad network as in Example 1 with $\mathcal{A}=\{1\}$ and $\mathcal{B}=\{1,2\}$. Then the sequence of moves is as follows: At stage 1, player 1 chooses $x_{1,1}$; at stage 2 both players choose simultaneously the action profile $(x_{1,2},x_{2,2})$. In particular, player 1 is free to revise his*

[23] We thank an anonymous referee for suggesting these possibilities.

earlier action $x_{1,1}$ to $x_{1,2}$ at stage 2. The payoff of player i is given by the profile of last actions chosen $u_i(x_{1,2}, x_{2,2})$.

The action space for each player is $\mathbb{R}_+$. A history at stage 2 is $h_2 = (x_{1,1})$. Player 1's strategy is $(s_{1,1}, s_{1,2}(x_{1,1}))$, where $s_{1,1} \in \mathbb{R}_+$, $s_{1,2} : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Player 2's strategy is $s_{2,2}(x_{1,1})$, where $s_{2,2} : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

Consider the subgame at history $h_2 = (x_{1,1})$. Note that both players' final payoffs do not directly depend on $x_{1,1}$. Hence $x_{1,1}$ is a payoff irrelevant signal that is observed by both players at stage 2. Therefore, dropping the stage subscript, the stage 2 subgame at every history is equivalent to the original simultaneous-move game with $(X_1 = \mathbb{R}_+, X_2 = \mathbb{R}_+, u_1, u_2)$, which has a unique NE

$$(b_1, b_2) = \left(\frac{a_1 + \delta a_2}{1 - \delta^2}, \frac{a_2 + \delta a_1}{1 - \delta^2} \right).$$

So (b_1, b_2) is the unique NE outcome in the subgame at each history $(x_{1,1})$. By backward induction, at stage 1, player 1 will choose an arbitrary $x_{1,1}$ knowing that this action is payoff irrelevant and it will not affect the NE outcome at stage 2. A SPNE of this modified game is $s_{1,1} \in \mathbb{R}_+$, $s_{1,2}(\cdot) = b_1$ and $s_{2,2}(\cdot) = b_2$.

Finally, in the two-stage game with overlapping target $(\mathcal{A}, \mathcal{B}) = (\{1\}, \{1, 2\})$, the equilibrium aggregate effort $b_1 + b_2$ is strictly lower than that in Example 1; hence, the designer is worse off here. Proposition 1 below also shows that both players are strictly worse off here.

To precisely describe a general two-stage game with revision $(\mathbf{G}, \delta, \mathcal{A}, \mathcal{B}; \mathbf{a})$, we introduce some more notation. A history at the beginning of stage 2 is $h_2 = (x_{i,1})_{i \in \mathcal{A}}$ that belongs to the set $H_2 = (X_i)_{i \in \mathcal{A}}$ (where $X_i = \mathbb{R}_+$). Histories are perfectly observable by all players in the game.[24] Hence, players' strategies are: (i) $s_i^1 \in \mathbb{R}_+$ if $i \in \mathcal{A} \backslash \mathcal{B}$; (ii) $s_i^2 : H_2 \mapsto \mathbb{R}_+$ if $i \in \mathcal{B} \backslash \mathcal{A}$; and (iii) (s_i^1, s_i^2) such that $s_i^1 \in \mathbb{R}_+, s_i^2 : H_2 \mapsto \mathbb{R}_+$ if $i \in \mathcal{A} \cap \mathcal{B}$. Denote by $\mathbf{s}$ some strategy profile in this game. Let $\tilde{\mathbf{s}}$ denote a SPNE of this game and let $\tilde{\mathbf{x}} = ((\tilde{x}_j^1)_{j \in \mathcal{A} \backslash \mathcal{B}}, (\tilde{x}_j^2)_{j \in \mathcal{B}})$ denote the last action chosen by each player in this equilibrium. Only the equilibrium outcome $\tilde{\mathbf{x}}$ is payoff relevant. Again the designer aims to maximize total equilibrium effort $\tilde{x} = \mathbf{1}'\tilde{\mathbf{x}}$.

The next result says that every game with overlapping target induces the same payoff-relevant equilibrium outcome as an equivalent baseline targeting game without revision (i.e., all the players are only allowed to move once when asked).

Lemma 4. *For any permissible two-stage game with overlapping target $(\mathbf{G}, \delta, \mathcal{A}, \mathcal{B}; \mathbf{a})$, the SPNE outcome*

$$\tilde{\mathbf{x}}(\mathbf{G}, \delta, \mathcal{A}, \mathcal{B}; \mathbf{a}) = \mathbf{x}(\mathbf{G}, \delta, \mathcal{A} \backslash \mathcal{B}; \mathbf{a}).$$

Intuitively, in the game $(\mathbf{G}, \delta, \mathcal{A}, \mathcal{B}; \mathbf{a})$ every player $i \in \mathcal{A} \cap \mathcal{B}$ is asked to move twice: First, he chooses action $x_{i,1}$ in stage 1 with all agents in $\mathcal{A}$; Second, he may revise this action to $x_{i,2}$

[24] As critically pointed out by Bagwell (1995), when histories are imperfectly observable with a small noise, the SPNE outcome in the sequential-move game can be equivalent to the NE in the simultaneous-move game – that is, the Stackleberg first-mover advantage may disappear.

in stage 2 after observing $(x_{j,1})_{j\in\mathcal{A}}$ from stage 1 and knowing that all agents in $\mathcal{A}\cap\mathcal{B}$ can revise their actions in stage 2. All actions $x_{i,1}$ taken in stage 1 by players in $\mathcal{A}\cap\mathcal{B}$ are simply cheap talk messages that they cannot commit to until the end.[25] Hence, in equilibrium all followers will ignore these actions and only best respond to the credible actions $(x_{j,1})_{j\in\mathcal{A}\backslash\mathcal{B}}$ chosen by players in $\mathcal{A}\backslash\mathcal{B}$. The SPNE outcome of the game with $\mathcal{A}$ lead and $\mathcal{B}$ follow is equivalent to that when the designer chooses to only let $\mathcal{A}\backslash\mathcal{B}$ lead and the rest, $\mathcal{B}$, follow.

Definition 3. *For each pair $(\mathcal{A},\mathcal{B})$ with $\mathcal{A}\cup\mathcal{B}\subseteq\mathcal{N}$, let $\tilde{\gamma}(\mathbf{G},\delta,\mathcal{A},\mathcal{B};\mathbf{a})$ be the unique positive scalar γ such that*

$$\tilde{x}(\mathbf{G},\delta,\mathcal{A},\mathcal{B};\mathbf{a}) = b(\mathbf{G},\gamma\delta;\mathbf{a}). \tag{11}$$

*We call $\tilde{\gamma}$ the **relative network synergy equivalent (RNSE)** of overlapping target $(\mathcal{A},\mathcal{B})$.*

Theorem 3. *For all permissible two-stage game with (possibly overlapping) target $(\mathcal{A},\mathcal{B})$,*

$$\tilde{\gamma}(\mathbf{G},\delta,\mathcal{A},\mathcal{B};\mathbf{a}) = \gamma(\mathbf{G},\delta,\mathcal{A}\backslash\mathcal{B};\mathbf{a})$$

Proof. It follows directly from Lemma 4 and the definition of $\tilde{\gamma}$. □

By virtue of Theorem 3, all our previous results about γ apply to $\tilde{\gamma}$. In particular, $\sqrt{2}$ is a tight upper bound of $\tilde{\gamma}$.

Remark 3. *In the game $(\mathbf{G},\delta,\mathcal{A},\mathcal{B};\mathbf{a})$, allowing a player $i\in\mathcal{A}\cap\mathcal{B}$ to move in both stages undermines the credibility of his earlier action $x_{i,1}$. When player i has the flexibility to revise his action in the later stage, he can no longer credibly commit to his earlier choice.*

When all agents can revise their decisions, that is, $\mathcal{B}=\mathcal{N}$, the game is outcome equivalent to the simultaneous-move game. In that case, $\tilde{\gamma}=1$ and targeting has no value. Clearly, the designer is worse off as the aggregate effort is lower than that in the baseline game $(\mathbf{G},\delta,\mathcal{A},\mathcal{N}\backslash\mathcal{A};\mathbf{a})$.

In addition, the next proposition shows that every player is hurt by the opportunity to revise his action. If the seeds have the option between delaying his decision and acting first, they would prefer acting first.

Proposition 1. *Compared with the game $(\mathbf{G},\delta,\mathcal{A},\mathcal{N}\backslash\mathcal{A};\mathbf{a})$, every player's equilibrium payoff is strictly lower in the game $(\mathbf{G},\delta,\mathcal{A},\mathcal{N};\mathbf{a})$.*

Proof. It follows directly from comparing parts (i) and (ii) of Lemma 5 in Appendix A.6. □

To see the intuition, consider the special case when there is only one seed player $\mathcal{A}=\{i\}$. For the seed player i, he could always choose his Nash equilibrium effort b_i in the simultaneous-move

[25]All other players know that players in $\mathcal{A}\cap\mathcal{B}$ cannot commit to their stage 1 action when given the opportunity to revise at stage 2, and players in $\mathcal{A}\cap\mathcal{B}$ know that other players know that they cannot commit and may revise action later if it is beneficial to do so.

game, and the other players will best respond in stage 2 by choosing their simultaneously-move Nash equilibrium effort b_{-i}. Thus, the simultaneously-move Nash equilibrium outcome (b_i, b_{-i}) is feasible for player i, but he can act first and achieve the strictly higher SPNE effort (x_i^S, x_{-i}^S). Since $\mathbf{x}^S$ is the unique SPNE, player i is strictly better off at outcome (x_i^S, x_{-i}^S) than (b_i, b_{-i}). Alternatively, if player i delays decision till stage 2, he only gets the NE outcome (b_i, b_{-i}). This echoes the familiar intuition for the Stackleberg first-mover advantage. In our setting, the followers also benefit from a higher action by the leader, due to strategic complementarity and positive spillovers.

The argument for the general case with non-singleton $\mathcal{A}$ is similar, but more involved. One caveat is that a leader $i \in \mathcal{A}$ cannot induce the NE outcome and payoff by deviating to b_i due to the presence of other leaders. See Lemma 5 in Appendix A.6 for details.

A variant of the above analysis is to only allow for $x_{i,2} \geq x_{i,1}$. In other words, targeted players can only increment their chosen actions in the later stage, which is often the case in crowdfunding or charity giving. This monotonicity constraint partially restores some commitment power of the first-mover(s). To illustrate, consider again the Dyad network in Example 4. Assume that at stage 2 player 1 can only revise upward his earlier action (i.e., $x_{1,2} \geq x_{1,1}$). Clearly, player 1 benefits strictly from choosing at stage 1 the higher SPNE action x_1^S rather than the equilibrium action b_1 in the no commitment case (or any other action). In fact, choosing $x_{1,2} = x_{1,1} = x_1^S$ is player 1's equilibrium action in this partial commitment case. Hence, the targeted players will choose not to increment in stage 2 – they are expected to moves first and indeed have every incentive to move first.

To summarize, if some node is granted the flexibility to revise action later, it undermines the credibility of his earlier action, rendering this action meaningless. If a targeted node can choose between having the flexibility to revise or giving up this option, the targeted node will surrender the option and act immediately in stage 1. The finding suggests that being targeted and moving in stage 1 complies with every player's self-interest (as well as the planner's goal), and the planner's arrangement will be voluntarily followed through. In other words, targeting leads to a Pareto improvement and consequently will be welcomed (and perhaps endorsed) by everyone in the network.

5 Extensions

5.1 Incorporating pricing

We consider a simple application of our results to a monopolist who sells network products to a group of users. This is related to a number of widely-studied problems such as product diffusion, referrals, and differential pricing for network goods, see, for e.g., Campbell (2013), Galeotti and

Goyal (2009), and Lobel et al. (2017). We adopt the following payoff

$$\pi_i(x_1, x_2, \cdots, x_n) = a_i x_i - \frac{1}{2}x_i^2 + \delta \sum_{j\in\mathcal{N}} g_{ij}x_i x_j - p_i x_i.$$

Here p_i is the per-unit price of user i. We assume that the firm can commit to the price vector $\mathbf{p} = (p_1, \cdots, p_n)$ in stage 0. Fixing the target $\mathcal{A}$, the users play the consumption stages sequentially as before, taking the prices as given. Let $\mathbf{X}(\mathbf{G}, \delta, \mathcal{A}, \mathbf{p})$ be the corresponding demand profile induced by the consumption subgame. The firm's maximal profit is equal to

$$\Pi^*(\mathbf{G}, \delta; \mathcal{A}) = \max_{\mathbf{p}} \langle \mathbf{p} - \mathbf{c}, \mathbf{X}(\mathbf{G}, \delta, \mathcal{A}, \mathbf{p})\rangle.$$

The benchmark case is $\mathcal{A} = \emptyset$ which generates profit $\Pi^*(\mathbf{G}, \delta; \emptyset)$.[26]

We can define γ^f in the same fashion as Definition 1 but from the perspective of the firm:

$$\Pi^*(\mathbf{G}, \delta; \mathcal{A}) = \Pi^*(\mathbf{G}, \gamma^f\delta; \emptyset).$$

Without loss of generality, we assume $\mathbf{c} = \mathbf{0}$ and $\mathbf{a} = \mathbf{1}_n$. The next theorem shows a strong mapping between aggregate action maximization and revenue maximization.

Theorem 4. *For any targeting strategy $\mathcal{A}$, $\gamma^f = \gamma$.*

By Theorem 4, all our previous results about γ apply to γ^f. In particular, the upper bound of γ applies to γ^f. So the optimal monopoly profit with optimally designed sequential launch with δ is dominated by the optimal profit with simultaneous launch with δ' as long as

$$\delta' \geq \sqrt{2}\delta.$$

Similar to our base model, one application of Theorem 4 is to compare two policies of a monopolist which sells social products to a group of consumers: (1) sequential launch, and (2) synergy-improving technology: improve δ to δ'. Because of the strong connection indicated in Theorem 4, the discussions and implications are all analogous in this revenue maximization context. Thus we omit them to avoid redundancy.

5.2 Costly seeding

In the baseline model, we assume that seeding is costless. Do our main results extend to the case with costly seeding? To answer the question, let $\rho(\mathcal{A}) \geq 0$ denote the cost of seeding $\mathcal{A} \subseteq \mathcal{N}$ and normalize $\rho(\emptyset) = 0$.[27] We incorporate the cost of seeding into the following definition of γ.

[26] Bloch and Quérou (2013) and Candogan et al. (2012) study the optimal pricing problem with simultaneous consumption ($\mathcal{A} = \emptyset$).

[27] One example is $\rho(\mathcal{A}) = t|A|$, where $t \geq 0$ is the constant marginal cost of seeding one node. Our results below regarding costly seeding (Proposition 2) does not depend on any specific functional form, or monotonicity of ρ.

Definition 4. *For each $\mathcal{A} \subseteq \mathcal{N}$, let $\gamma_\rho(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ be the unique positive scalar γ such that*

$$x(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) - \rho(\mathcal{A}) = b(\mathbf{G}, \gamma\delta; \mathbf{a}). \tag{12}$$

*We call γ_ρ the **relative network synergy equivalent (RNSE)** of targeting $\mathcal{A}$ with costly seeding.*

Note that in the above definition, the benchmark is adjusting δ while not using any seeding; i.e., $\mathcal{A} = \emptyset$, in which case $\rho(\emptyset) = 0$. Since seeding is costly, it is easy to see that $\gamma_\rho(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \leq \gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) < \sqrt{2}$.

Moreover, $\sqrt{2}$ is also a tight upper bound for γ_ρ. To see this, recall the Dyad network from Example 1 with $\mathcal{A} = \{1\}$ and $a_1 = a_2 = 1$. By definition, $\gamma_\rho(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ solves the following equation

$$\frac{2(1+\delta) - \delta^2}{1 - 2\delta^2} - \rho(\{1\}) = 2\frac{1 + \gamma_\rho\delta}{1 - (\gamma_\rho)^2\delta^2}. \tag{13}$$

The range of δ is $[0, 1/\sqrt{2})$. Note that as δ approaches $1/\sqrt{2}$, γ_ρ approaches $\sqrt{2}$. This is because the left-hand side of (13) approaches to infinity as $\delta \to 1/\sqrt{2}$, and therefore $\lim_{\delta \to 1/\sqrt{2}}(1 - (\gamma_\rho)^2\delta^2) = 0$. Consequently, $\lim_{\delta \to 1/\sqrt{2}} \gamma_\rho = \sqrt{2}$.

Summarizing the above discussions, we can show that both of our main findings in the baseline model remain valid under costly seeding.

Proposition 2. *With the presence of costly seeding, the following results hold.*

(i) For any permissible two-stage network game $(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$,

$$\gamma_\rho(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) < \sqrt{2}.$$

(ii) For any $\epsilon > 0$, there exists a permissible two-stage network game $(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ with

$$\gamma_\rho(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) > \sqrt{2} - \epsilon.$$

Remark 4. *The above definition of γ_ρ is from the perspective of a designer maximizing the aggregate effort. Analogously, we could define an index, called γ_ρ^f, from the perspective of a monopoly firm maximizing revenue as introduced in Section 5.1:*

$$\Pi^*(\mathbf{G}, \delta; \mathcal{A}) - \rho(\mathcal{A}) = \Pi^*(\mathbf{G}, \gamma_\rho^f\delta; \emptyset),$$

where ρ is interpreted as the monetary cost of seeding. It is straightforward to see that our results regarding costly seeding (Proposition 2) hold if we replace γ_ρ by γ_ρ^f.

5.3 k-random seeding

In related recent work (Akbarpour et al. 2020; Beaman et al. 2021), optimal seeding is often compared with random seeding (of the same size). To illustrate our findings in this context, suppose now the benchmark is randomly seeding k nodes (instead of no seeding $\mathcal{A} = \emptyset$). We can redefine the RNSE index as follows.

Definition 5. *For each $\mathcal{A} \subseteq \mathcal{N}$, let $\gamma_k(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ be the unique positive scalar γ such that*[28]

$$x(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) = \mathbb{E}_k[x(\mathbf{G}, \gamma\delta, \mathcal{B}; \mathbf{a})], \tag{14}$$

where $\mathcal{B}$ is a random subset with k nodes and the expectation $\mathbb{E}$ is taken for $\mathcal{B}$. We call γ_k the ***relative network synergy equivalent (RNSE)*** *of targeting $\mathcal{A}$ with random k-seeding.*

Obviously, when $k = 0$, the benchmark is no seeding. The index γ_0 is the same as our RNSE index γ proposed in Definition 1.

When $k \geq 1$, γ_k defined above can be smaller than one, whereas γ is always weakly greater than one. Comparing (14) and (5) (in Definition 1), we observe that the benchmark value is higher with random k-seeding, implying that the corresponding index γ_k is always weakly lower than γ.

Proposition 3. *For any permissible two-stage network game $(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$,*

$$\gamma_k(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) < \sqrt{2}.$$

Proof. Because seeding is always valuable, $x(\mathbf{G}, \gamma\delta, \mathcal{B}; \mathbf{a}) \geq b(\mathbf{G}, \gamma\delta; \mathbf{a})$ for any $\mathcal{B}$. Taking expectation yields $\mathbb{E}_k[x(\mathbf{G}, \gamma\delta, \mathcal{B}; \mathbf{a})] \geq b(\mathbf{G}, \gamma\delta; \mathbf{a})$, which implies that $\gamma_k(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \leq \gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$. □

Hence, $\sqrt{2}$ remains an upper bound for γ_k when the benchmark is k-random seeding.

5.4 Multi-stage sequences

We consider the general sequential launch problem with multiple stages of moves. To set up the problem, we first define a multi-stage sequence and its refinement.

A *sequence* $\mathcal{S} = (P_1, P_2, \cdots, P_k)$ is a partition of $\mathcal{N}$ such that $P_i \cap P_j = \emptyset, \forall i \neq j$ and $\cup_{1 \leq i \leq k} P_i = \mathcal{N}$, where $|\mathcal{S}| = k$ corresponds to the number of steps of this sequence. A *chain* is a sequence with N step; i.e., players determine their contributions one by one. In addition, a sequence $\mathcal{S}'$ is *a refinement of* $\mathcal{S} = (P_1, P_2, \cdots, P_k)$ if there exists $r = 1, \ldots, k$ such that $\mathcal{S}' = (P_1, \cdots, P_{r-1}, Q_1, Q_2, P_{r+1}, \cdots P_k)$ where $Q_1 \cup Q_2 = P_r$ and $Q_1 \cap Q_2 = \emptyset$.

Given $\mathbf{G}, \delta, \mathcal{S}$, we can define an *extensive-form* game with complete information in which $\mathcal{N}$ players move according to the sequence specified in $\mathcal{S}$. The timing is as follows. Players in P_1 move simultaneously in the first period, and then players in P_2 move simultaneously in the second

[28] With random seeding benchmark, the permissible range of δ need to be adjusted accordingly.

period, $\cdots$, and players in P_k move simultaneously in the k-th period. The actions, once taken, are observable to all remaining players who move later. Let $\mathbf{x}(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a})$ be the SPNE effort profile in the multi-stage game and let $x(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a}) = \mathbf{1}'\mathbf{x}(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a})$ be the aggregate equilibrium effort. Analogously, we can define the effectiveness index for the sequential targeting policy $\mathcal{S}$.

Definition 6. *For network* $\mathbf{G}$*, synergy* δ*, marginal utility vector* $\mathbf{a}$*, and sequence* $\mathcal{S}$*, let* $\gamma(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a})$ *be unique positive scalar such that*

$$x(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a}) = b(\mathbf{G}, \gamma\delta; \mathbf{a}).$$

This γ *index is called the* relative network synergy equivalent (RNSE) for sequence $\mathcal{S}$.

The regular RNSE obtains when the sequence has only one set of agents moving first and the remaining moving second, i.e., $k = 2$. For fixed $\mathbf{G}, \delta$, the designer's objective is $\max_{\mathcal{S}} x(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a})$, which is equivalent to $\max_{\mathcal{S}} \gamma(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a})$.

Proposition 4. *Fix any* $\mathbf{G}$ *and* $\delta \in \left[0, \frac{1}{\sqrt{2}\lambda_{max}(\mathbf{G})}\right)$.

1. *If* $\mathcal{S}''$ *is a refinement of* $\mathcal{S}'$*, then* $\gamma(\mathbf{G}, \delta, \mathcal{S}''; \mathbf{a}) > \gamma(\mathbf{G}, \delta, \mathcal{S}'; \mathbf{a})$.
2. *For any sequence* $\mathcal{S}$*,* $\gamma(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a}) < \sqrt{2}$.

Proposition 7 in Zhou and Chen (2015) shows that equilibrium effort increases with a refinement of the sequence $\mathcal{S}$. Part (i) of Proposition 4 is a direct corollary of their observation and the definition of index γ. More importantly, part (ii) says that allowing for more than two rounds of sequential launches cannot improve the upper bound of γ further than the uniform bound of $\sqrt{2}$ shown by Theorem 1. The proof is analogous to that of Theorem 1, using an induction argument on the depth of the sequence $\mathcal{S}$.

In other words, further refinement of the sequence of targeting has limited value. This observation does not reject the usefulness of refinement. In fact, Zhou and Chen (2015) show that refinement is always strictly beneficial to the designer, as suggested by part (i) of Proposition 4. As a result, having a chain is always optimal (as long as having more stages is without cost). Nevertheless, Proposition 4 (ii) says that the value of having a chain is still bounded by $\sqrt{2}$.

Combing these observations, we have a more complete understanding about the benefit of sequentiality in network games.

Corollary 1. *Suppose* $\delta \in \left[0, \frac{1}{\sqrt{2}\lambda_{\max}(\mathbf{G})}\right)$*. Some SPNE exists and it is unique for any sequence* $\mathcal{S}$*. The value of sequentiality is limited as the* γ *is less than* $\sqrt{2}$*. Moreover, the SPNE outcome is lower than the first best.*

A concern similar to Section 4.5 is that the designer may not have precise control over the exact sequence of moves. Some nodes may have the opportunity to delay or revise their actions later. Similarly, we can extend the multi-stage game and our index γ to allow for revisions.

A *generalized sequence* $\tilde{\mathcal{S}} = (\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_k)$ is a sequence of subsets of $\mathcal{N}$ such that $\cup_{1\leq i\leq k}\tilde{P}_i = \mathcal{N}$. Here $\tilde{P}_i$ and $\tilde{P}_j$ may not be disjoint. Similarly, we can define a k-stage extensive-form game with complete information, in which players move according to the sequence in $\tilde{\mathcal{S}}$. Similar to Section 4.5, we assume only the final action chosen by each node is payoff relevant; i.e., for all i, $u_i(\cdot)$ is a function of $\mathbf{x} := (x_{1,\tau(1)}, \ldots, x_{n,\tau(n)})$, where $\tau(i)$ is the last stage i can move and $x_{i,\tau(i)}$ is the last action taken by player i in $\tilde{\mathcal{S}}$. The sequential game $(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a})$ is described in detail in Appendix A.10. Let $\tilde{\mathbf{s}}$ be a SPNE of this game, and let $\tilde{\mathbf{x}}(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a})$ be the payoff relevant SPNE outcome. For each $\tilde{\mathcal{S}}$, define a partitional sequence $\mathcal{S} = (P_1, P_2, \cdots, P_k)$ that assigns every node i to the last stage he is asked to move; i.e., $P_t := (\cup_{s\geq t}\tilde{P}_s)\backslash(\cup_{s\geq t+1}\tilde{P}_s)$ for all $t = 1, \ldots, k$ (with $\tilde{P}_{k+1} = \emptyset$). Lemma 6 in Appendix A.10 shows that every k-stage network game with generalized sequence $(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a})$ is *outcome equivalent* to a k-stage game with a corresponding partitional sequence $(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a})$.

We can define the index $\tilde{\gamma}$ for sequential game with overlapping sequence $\tilde{\mathcal{S}}$ in the same fashion as Definition 1:

$$\tilde{x}(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a}) = b(\mathbf{G}, \tilde{\gamma}\delta; \mathbf{a}).$$

Proposition 5. *For any permissible k-stage network game $(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a})$,*

$$\tilde{\gamma}(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a}) = \gamma(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a}).$$

Proof. It follows directly from Lemma 6 and definition of $\tilde{\gamma}$. □

Proposition 5 suggests that value of targeting by $\tilde{\mathcal{S}}$, as measured by the RNSE index $\tilde{\gamma}$, can always be achieved in a game where the designer chooses a partitional sequence $\mathcal{S}$. As a result, focusing on the main case where the designer lets everyone move exactly once is *without loss of generality.*

6 Conclusion

It is widely accepted that sequential launch of network goods with targeted users improves aggregate action in the contexts of charity donation, public good provision, teamwork production, technology adoption. In this paper, we show that for any network structure, the equilibrium consumption with any selection of targets/seeds is dominated by the outcome in which the peer effect parameter is multiplied by $\sqrt{2} \approx 1.414$. In other words, about 41.4% increase in the magnitude of the peer effect is the upper bound on the value of any targeting policy. We also demonstrate the tightness of this bound $\sqrt{2}$. Therefore, even though it is valuable to use targeting based on network information, the value of such network-based targeting may be rather limited. We further identify scenarios in which the benefit from the optimal targeting is negligible. This analysis allows policy makers to evaluate the cost of investing in network technology versus the cost of network information.

We extend our results along various dimensions: the pricing of a monopolist selling social products, cost of intervening, randomly seeding a fixed number of players as an alternative benchmark, and the general sequential problem with multiple stages of moves. We show that $\sqrt{2}$ remains the (uniform) upper bound for all the above extensions. As another robustness check, we consider the extension when the designer lets some seed players to revise their actions at the later stage. We find that if a targeted node can choose between having the flexibility to revise or giving up this option, the targeted node will surrender the option. We also consider a variant in which targeted players can only increment their chosen actions in the later stage; in this case, all targeted players will choose not to increment in the later stage. In both scenarios, being targeted and moving in the early stage complies with the players' self-interest, and the planner's arrangement will be voluntarily executed.

Our measure of RNSE has two appealing features. First, the measure is invariant to any affine transformation on the payoffs of players. This property is desirable because in most expected utility models payoff functions are typically only identified up to positive affine transformations. Hence robust economic predictions shall not be altered when we scale these payoffs. This invariance no longer holds if we consider alternative measures such as absolute change in the aggregate action or payoff. Second, it enables us to draw meaningful comparison across different targeting strategies, and across different network structures.

Our efforts hopefully invite further investigations of such an exciting area. Within the same framework, it might be sensible to identify sharp upper and lower bounds for specific network $\mathbf{G}$ and specific sequence $\mathcal{S}$.[29] Extensions to weighted networks appear straightforward, but may be of independent interest. Beyond our setting, the idea of quantifying the value of network information and network tactics can be applied to studying other types of targeting and network models with alternative functional forms and/or action space (discrete binary choice vs continuous choice). How to extend our analysis to those settings? Is there any constant performance bound for the value of optimal targeting? These remain a research priority.

[29] See Section B.2 in Supplementary Materials for some examples.

A Appendix: Proofs and more results

A.1 Proof of Theorem 1

The inverse Leontief matrix $[\mathbf{I}-\delta\mathbf{D}]^{-1}$ associated with any matrix $\mathbf{D}$ can be expressed by the following summation

$$[\mathbf{I}-\delta\mathbf{D}]^{-1} = \mathbf{I} + \delta\mathbf{D} + \delta^2\mathbf{D}^2 + \cdots, \tag{15}$$

where the infinite sum on the right hand side converges provided the spectral radius of $\delta\mathbf{D}$ is less than 1, in particular it holds for a sufficiently small $\delta > 0$. Using (15), we can show two intuitive properties of the inverse Leontief matrix. (1) If $\mathbf{D} \succsim (\succ)\mathbf{0}$, then $[\mathbf{I}-\delta\mathbf{D}]^{-1} \succsim (\succ)\mathbf{I} \succ \mathbf{0}$; (2) If $\mathbf{D}' \succsim (\succ)\mathbf{D}$, then $[\mathbf{I}-\delta\mathbf{D}']^{-1} \succsim (\succ)[\mathbf{I}-\delta\mathbf{D}]^{-1}$. We will use them in the proof.

Fix any network $\mathbf{G}$. Consider any two-stage game with players in $\mathcal{A} \subseteq \mathcal{N}$ move first followed by players in $\mathcal{B} = \mathcal{N}\backslash\mathcal{A}$. We express $\mathbf{G} = \begin{pmatrix} \mathbf{G}_{AA} & \mathbf{G}_{AB} \\ \mathbf{G}_{BA} & \mathbf{G}_{BB} \end{pmatrix}$ and denote $\mathbf{U}(\delta) = [\mathbf{I}-\delta\mathbf{G}_{BB}]^{-1}$.

By Proposition 1 in Zhou and Chen (2015), the SPNE action vector in the two-stage game can be expressed as (omitting dependence on $\mathbf{G}$ in notation)

$$\mathbf{x}^S(\delta) = \mathbf{Z}(\delta)\mathbf{a},$$

where $\mathbf{Z}(\delta) := \begin{pmatrix} \mathbf{Z}(\delta)_{AA} & \mathbf{Z}(\delta)_{AB} \\ \mathbf{Z}(\delta)_{BA} & \mathbf{Z}(\delta)_{BB} \end{pmatrix}$ is given by

$$\begin{pmatrix} \left[\mathbf{I}-\delta(\mathbf{T}(\delta)+\mathbf{T}(\delta)^D)\right]^{-1} & \delta\left[\mathbf{I}-\delta(\mathbf{T}(\delta)+\mathbf{T}(\delta)^D)\right]^{-1}\mathbf{G}_{AB}\mathbf{U}(\delta) \\ \delta\mathbf{U}(\delta)\mathbf{G}_{BA}\left[\mathbf{I}-\delta(\mathbf{T}(\delta)+\mathbf{T}(\delta)^D)\right]^{-1} & \mathbf{U}(\delta)+\delta^2\mathbf{U}(\delta)\mathbf{G}_{BA}\left[\mathbf{I}-\delta(\mathbf{T}(\delta)+\mathbf{T}(\delta)^D)\right]^{-1}\mathbf{G}_{AB}\mathbf{U}(\delta) \end{pmatrix}, \tag{16}$$

with $\mathbf{T}(\delta) = \mathbf{G}_{AA} + \delta\mathbf{G}_{AB}\,\mathbf{U}(\delta)\,\mathbf{G}_{BA}$.

Recall the simultaneous-move game has equilibrium effort

$$\mathbf{x}(\delta) = [\mathbf{I}-\delta\mathbf{G}]^{-1}\mathbf{a} := \mathbf{M}(\delta)\mathbf{a} := \mathbf{b}(\delta).$$

We prove a stronger statement than Theorem 1: for all permissible triple $(\mathbf{G}, \delta, \mathcal{A})$,

$$(\star) \qquad \mathbf{Z}(\delta) \prec \mathbf{M}(\sqrt{2}\delta). \tag{17}$$

To see this, recall that in the proof of Proposition 1 in Zhou and Chen (2015), the inverse Leontief matrix $\mathbf{M}(\sqrt{2}\delta)$ can be expressed as $\begin{pmatrix} \mathbf{M}(\sqrt{2}\delta)_{AA} & \mathbf{M}(\sqrt{2}\delta)_{AB} \\ \mathbf{M}(\sqrt{2}\delta)_{BA} & \mathbf{M}(\sqrt{2}\delta)_{BB} \end{pmatrix}$, which equals

$$\begin{pmatrix} \left[\mathbf{I}-\sqrt{2}\delta(\mathbf{T}(\sqrt{2}\delta)+\mathbf{0})\right]^{-1} & \sqrt{2}\delta\left[\mathbf{I}-\sqrt{2}\delta(\mathbf{T}(\sqrt{2}\delta)+0)\right]^{-1}\mathbf{G}_{AB}\mathbf{U}(\sqrt{2}\delta) \\ \sqrt{2}\delta\mathbf{U}(\sqrt{2}\delta)\mathbf{G}_{BA}\left[\mathbf{I}-\sqrt{2}\delta(\mathbf{T}(\sqrt{2}\delta)+0)\right]^{-1} & \mathbf{U}(\sqrt{2}\delta)+2\delta^2\mathbf{U}(\sqrt{2}\delta)\mathbf{G}_{BA}\left[\mathbf{I}-\sqrt{2}\delta(\mathbf{T}(\sqrt{2}\delta)+0)\right]^{-1}\mathbf{G}_{AB}\mathbf{U}(\sqrt{2}\delta) \end{pmatrix}. \tag{18}$$

To show equation (17), we need to show the matrix expressed by (18) dominates the matrix expressed by (16) entry by entry. To do so, we compare the two matrices block by block using the following observations:

- $\mathbf{G}_{AB} \gtrsim 0$, $\mathbf{G}_{BA} \gtrsim 0$.
- For sufficiently small $\delta > 0$, $\mathbf{U}(\delta) = [\mathbf{I} - \delta \mathbf{G}_{BB}]^{-1} \gtrsim \mathbf{I} \succ \mathbf{0}$, and $\mathbf{U}(\sqrt{2}\delta) \succ \mathbf{U}(\delta)$. They follow from $\mathbf{G}_{BB} \gtrsim \mathbf{0}$ and the two properties of the inverse Leontief matrix mentioned above.
- $\mathbf{T}(\delta) = \mathbf{G}_{AA} + \delta \mathbf{G}_{AB}\, \mathbf{U}(\delta)\, \mathbf{G}_{BA} \gtrsim \mathbf{0}$. Similarly, $\mathbf{T}(\sqrt{2}\delta) \gtrsim \mathbf{0}$. By the same argument, the inverse Leontief matrices $[\mathbf{I} - \delta(\mathbf{T}(\delta) + \mathbf{T}(\delta)^D)]^{-1} \succ \mathbf{0}$ and $[I - \sqrt{2}\delta \mathbf{T}(\sqrt{2}\delta)]^{-1} \succ \mathbf{0}$ for δ small enough.

Given these observations, to show equation (17), it suffices to show

$$[\mathbf{I} - \sqrt{2}\delta \mathbf{T}(\sqrt{2}\delta) + \mathbf{0}]^{-1} \succ [\mathbf{I} - \delta(\mathbf{T}(\delta) + \mathbf{T}(\delta)^D)]^{-1}. \tag{19}$$

So equation (19) is equivalent to

$$[\mathbf{I} - \delta(\mathbf{T}(\delta) + \mathbf{T}(\delta)^D)]^{-1} \cdot [\sqrt{2}\delta \mathbf{T}(\sqrt{2}\delta) - \delta(\mathbf{T}(\delta) + \mathbf{T}(\delta)^D)] \cdot [\mathbf{I} - \sqrt{2}\delta \mathbf{T}(\sqrt{2}\delta)]^{-1} \succ \mathbf{0}.$$

Here we use the property $\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{B}^{-1}$ for any matrices $\mathbf{A}$, $\mathbf{B}$.

Hence, to prove equation (19), it suffices to show

$$(**) \qquad \sqrt{2}\mathbf{T}(\sqrt{2}\delta) \succ \mathbf{T}(\delta) + \mathbf{T}^D(\delta). \tag{20}$$

Substituting the expression of $\mathbf{T}(\cdot)$ into equation (20), we need to show

$$\begin{aligned} &\sqrt{2}\mathbf{G}_{AA} + 2\delta \mathbf{G}_{AB}[\mathbf{I} - \sqrt{2}\delta \mathbf{G}_{BB}]^{-1}\mathbf{G}_{BA} \\ &\gtrsim \mathbf{G}_{AA} + \mathbf{G}_{AA}^D + \delta \mathbf{G}_{AB}[\mathbf{I} - \delta \mathbf{G}_{BB}]^{-1}\mathbf{G}_{BA} + \delta[\mathbf{G}_{AB}[\mathbf{I} - \delta \mathbf{G}_{BB}]^{-1}\mathbf{G}_{BA}]^D. \end{aligned} \tag{21}$$

Observe that:

- $\mathbf{G}_{AA} \gtrsim \mathbf{0}$ and $\mathbf{G}_{AA}^D = \mathbf{0}$ as the matrix $\mathbf{G}_{AA}$ has zero diagonal entries;
- For sufficiently small δ, $[\mathbf{I} - \sqrt{2}\delta \mathbf{G}_{BB}]^{-1} \gtrsim [\mathbf{I} - \delta \mathbf{G}_{BB}]^{-1} \gtrsim \mathbf{I}$. The claim follows from properties of the inverse Leontief matrix and $\delta > 0$, $\mathbf{G}_{BB} \gtrsim \mathbf{0}$;
- Since $\mathbf{G}_{AB}, \mathbf{G}_{BA}, [\mathbf{I} - \delta \mathbf{G}_{BB}]^{-1} \gtrsim \mathbf{0}$, we have $\mathbf{G}_{AB}[\mathbf{I} - \delta \mathbf{G}_{BB}]^{-1}\mathbf{G}_{BA} \gtrsim (\mathbf{G}_{AB}[\mathbf{I} - \delta \mathbf{G}_{BB}]^{-1}\mathbf{G}_{BA})^D \gtrsim \mathbf{0}$.

The three observations imply that equation (21) indeed holds. $\square$

A.2 Proof of Theorem 2

Consider the dyad example in Section 3.1. We have already shown that the index γ in (9) can be arbitrarily close to the upper bound $\sqrt{2}$ when δ approaches $1/\sqrt{2}$ (see Figure 1). $\square$

A.3 Proof of Lemma 3

The difference of the aggregate actions between the sequential-move game and the simultaneous-move one with targeting $\mathcal{A}$, when $a_i = 1$ for all i, equals

$$x(\mathbf{G}, \delta; \mathcal{A}) - b(\mathbf{G}, \delta) = \delta^2 \sum_{i \in \mathcal{A}} \sum_{j \in N \setminus \mathcal{A}} g_{ij} + \mathcal{O}(\delta^3)$$

by Zhou and Chen (2015). In addition,

$$b(\mathbf{G}, \delta) = \mathbf{1}'(\mathbf{I} + \sum_{k=1}^{\infty} \delta^k \mathbf{G}^k)\mathbf{1} = n + \delta \mathbf{1}'\mathbf{G}\mathbf{1} + \delta^2 \mathbf{1}'\mathbf{G}^2\mathbf{1} + \mathcal{O}(\delta^3).$$

From the definition of $\gamma(\mathbf{G}, \delta; \mathcal{A})$, we obtain that

$$n + \delta \mathbf{1}'\mathbf{G}\mathbf{1} + \delta^2 \mathbf{1}'\mathbf{G}^2\mathbf{1} + \mathcal{O}(\delta^3) + \sum_{i \in \mathcal{A}} \sum_{j \in N \setminus \mathcal{A}} g_{ij}\delta^2 = n + (\gamma\delta)\mathbf{1}'\mathbf{G}\mathbf{1} + (\gamma\delta)^2 \mathbf{1}'\mathbf{G}^2\mathbf{1} + \mathcal{O}(\delta^3),$$

which leads to

$$\gamma(\mathbf{G}, \delta; \mathcal{A}) = 1 + \left(\frac{\sum_{i \in \mathcal{A}} \sum_{j \in N \setminus \mathcal{A}} g_{ij}}{\mathbf{1}'\mathbf{G}\mathbf{1}} \right) \delta + \mathcal{O}(\delta^2) = 1 + \left(\frac{\mathsf{CUT}(\mathcal{A})}{\mathbf{1}'\mathbf{G}\mathbf{1}} \right) \delta + \mathcal{O}(\delta^2).$$

Clearly, $\gamma(\mathbf{G}, \delta; \mathcal{A})$ increases with δ when δ is relatively small. $\square$

A.4 Calculations for Example 3

Suppose $n = 3$ and $\mathcal{A} = \{1\}$. We compute γ as a function of δ for complete, star, and extended Dyad networks, respectively.

Complete network. We use $n = 3$ and $m = 1$ and obtain

$$\gamma^c(\delta) := \gamma(\delta, 3, 1) = \frac{3 + \delta - 6\delta^2}{3 - 7\delta^2},$$

where the range of permissible δ is $\left[0, \frac{2}{1+\sqrt{17}}\right) = [0, 0.1952)$.[30]

[30] In fact, $\mathcal{A} = \{1\}$ is also the optimal target group in the complete network. To see this, note that

$$\gamma(\delta, 3, 2) = \frac{3 + 4\delta}{3 + 3\delta - \delta^2}.$$

By tedious calculations, one can show $\gamma(\delta, 3, 1) - \gamma(\delta, 3, 2) > 0$. Thus, having one leader is optimal.

Star network. For a star network with 3 players and player 1 is the central node. By Zhou and Chen (2015), it is optimal to have the central player move first ($\mathcal{A} = \{1\}$).

In the simultaneous-move game with scaled synergy $\gamma\delta$, the players' best responses are

$$x_1 = 1 + 2\gamma\delta x_2, \qquad x_2 = x_3 = 1 + \gamma\delta x_1.$$

The NE action profile is

$$b_1(\gamma\delta) = \frac{1 + 2\gamma\delta}{1 - 2(\gamma\delta)^2}, \qquad b_2(\gamma\delta) = b_3(\gamma\delta) = 1 + \gamma\delta x_1 = \frac{1 + \gamma\delta}{1 - 2(\gamma\delta)^2}.$$

In the sequential-move game with synergy δ where player 1 moves first and players 2 and 3 follow. Player 1's payoff is

$$x_1 - \frac{1}{2}x_1^2 + 2\delta x_1(1 + \delta x_1).$$

The first-order condition is

$$1 - x_1 + 2\delta + 4\delta^2 x_1 = 0.$$

It implies the SPNE action profile is

$$\begin{aligned} x_1^S &= \frac{1 + 2\delta}{1 - 4\delta^2}, \\ x_2^S = x_3^S &= 1 + \delta x_1^S = 1 + \delta\left(\frac{1 + 2\delta}{1 - 4\delta^2}\right) = \frac{1 - 2\delta^2 + \delta}{1 - 4\delta^2}. \end{aligned}$$

By definition, γ is the scaler given by

$$\frac{(1 + 2\delta)^2}{1 - 4\delta^2} = x^S(\delta) = b(\gamma\delta) = \frac{(1 + 2\gamma\delta)^2}{1 - 2\gamma^2\delta^2},$$

which simplifies to

$$\gamma^2\delta(3 - 2\delta) + \gamma 2(1 - 2\delta) - 2 = 0.$$

The positive root is

$$\gamma^*(\delta) = \frac{-(1 - 2\delta) + \sqrt{(1 - 2\delta)^2 + 2\delta(3 - 2\delta)}}{\delta(3 - 2\delta)}.$$

Extended Dyad network. In this case, we have $n = 3$, $g_{12} = g_{21} = 1$ and $g_{ij} = 0$ for all $(i, j) \neq (1, 2), (2, 1)$. Note that player 3 is not connected to anyone. In both simultaneous-move and sequential-move games, player 3's optimal action is always $x_3(\delta) = b_3(\gamma\delta) = 1$ for all synergy parameter δ and scaler γ. Players 1 and 2 are interconnected and only interact with each other. So the strategic interaction between players 1 and 2 is the same as that in the two-person Dyad network analyzed before. And by definition, γ is given by equation (9)

$$\gamma^D(\delta) = \frac{2 + 3\delta}{2 + 2\delta - \delta^2},$$

for all permissible $\delta \in \left[0, \frac{1}{\sqrt{2}}\right)$.

A.5 Proof of Lemma 4

The game $(\mathbf{G}, \delta, \mathcal{A}, \mathcal{B}; \mathbf{a})$ with overlapping target has some SPNE $\tilde{\mathbf{s}}$ by permissibility and backward induction.

At stage 2, the subgame at history $h_2 = (x_{i,1})_{i \in \mathcal{A}}$ is a simultaneous-move game among players $\mathcal{B}$ with payoffs

$$u_i(\mathbf{x}_{B,2}; \mathbf{x}_{A \backslash B,1}) = \left(a_i + \delta \sum_{j \in \mathcal{A} \backslash \mathcal{B}} g_{ij} x_{j,1} \right) x_{i,2} - \frac{1}{2}(x_{i,2})^2 + \delta \sum_{j \in \mathcal{B}} g_{ij} x_{i,2} x_{j,2} \tag{22}$$

for all $i \in \mathcal{B}$. Note that players in $\mathcal{B}$ knows that only action profile $\mathbf{x}_{A \backslash B} = (x_{j,1})_{j \in \mathcal{A} \backslash \mathcal{B}}$ are committed and credible; while players in $\mathcal{A} \cap \mathcal{B}$ can still revise their actions in stage 2. By permissibility, this subgame has a unique NE. Every player $i \in \mathcal{B}$ has a unique best response function $x_i(\mathbf{x}_{A \backslash B})$.

At stage 1, player $i \in \mathcal{A} \backslash \mathcal{B}$ anticipates best responses $(x_j(\mathbf{x}_{A \backslash B,1}))_{j \in \mathcal{B}}$ and chooses $x_{i,1}$ to maximize

$$u_i(\mathbf{x}_{A \backslash B,1}) = a_i x_{i,1} - \frac{1}{2}(x_{i,1})^2 + \delta \sum_{j \in \mathcal{A} \backslash \mathcal{B}} g_{ij} x_{i,1} x_{j,1} + \delta \sum_{j \in \mathcal{B}} g_{ij} x_{i,1} x_j(\mathbf{x}_{A \backslash B,1}). \tag{23}$$

Let $x_i(x_{A \backslash B,-i,1})$ be player i's best response function to other players' action in $\mathcal{A} \backslash \mathcal{B}$. Denote by $(\tilde{x}_{j,1})_{j \in \mathcal{A} \backslash \mathcal{B}}$ the unique fixed point of these best response functions.

Every player i in $\mathcal{A} \cap \mathcal{B}$ is indifferent among his first-stage action $x_{i,1}$, knowing that it is payoff irrelevant and it will not affect the NE of the stage-2 subgames.

In the extended game, the payoff relevant equilibrium outcome is $\tilde{\mathbf{x}} = ((\tilde{x}_{j,1})_{j \in \mathcal{A} \backslash \mathcal{B}}, (x_j(\tilde{\mathbf{x}}_{\mathcal{A} \backslash \mathcal{B},1}))_{j \in \mathcal{B}})$ are characterized as above. Observe that the conditions are the same as the equilibrium characterization of the standard game $(\mathbf{G}, \delta, \mathcal{A} \backslash \mathcal{B}; \mathbf{a})$ studied in Zhou and Chen (2015), where the leaders are $\mathcal{A} \backslash \mathcal{B}$ and the followers are $\mathcal{B}$. The result follows.

□

A.6 Lemma 5 and its proof

To understand the incentives of seed players, we need to compare their equilibrium payoffs, which are conveniently summarized in the following Lemma.

Lemma 5. *The following statements regarding equilibrium payoffs hold:*

(i) In the simultaneous move game without seeding, each agent k's equilibrium payoff is

$$\pi_k^*(\mathbf{b}(\mathbf{G}, \delta; \mathbf{a})) = \frac{1}{2}(b_k(\mathbf{G}, \delta, \mathbf{a}))^2,$$

where $b_k(\mathbf{G}, \delta, \mathbf{a})$ is k's equilibrium action in the state NE.

(ii) In the the two-stage game with seeding $\mathcal{A}$, at the unique SPNE outcome $\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$, for each $j \in \mathcal{B} = \mathcal{N} \backslash \mathcal{A}$, her equilibrium payoff is

$$\pi_j^*(\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})) = \frac{1}{2}(x_j(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}))^2 > \frac{1}{2}(b_j(\mathbf{G}, \delta, \mathbf{a}))^2;$$

while, for $i \in \mathcal{A}$, her equilibrium payoff satisfies

$$\pi_i^*(\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})) < \frac{1}{2}(x_i(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}))^2.$$

However, this payoff is strictly higher than what i obtains at the state NE, i.e.,

$$\pi_i^*(\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})) > \frac{1}{2}(b_i(\mathbf{G}, \delta, \mathbf{a}))^2.$$

Proof of Lemma 5. We will repetitively use the simple mathematical observation from the solution of a quadratic optimization problem: $f(z) = az - \theta z^2$ with $\theta > 0$. The unique maximizer is obtained at $z^* = a/(2\theta)$ with maximum value $\max_z f(z) = az^* - \theta(z^*)^2 = \theta(z^*)^2$.

Applying this observation to the static NE outcome and the payoff specification in (1), we note that the equilibrium payoff under NE outcome is precisely $\frac{1}{2}b_k^2$ as the equilibrium action of k is precisely b_k. This shows part (i) of Lemma 5.

Now we consider the two-stage game with seeding $\mathcal{A}$. First, for any $j \in \mathcal{B}$ who moves in the 2nd stage, j's equilibrium payoff is just $(1/2)(x_j^*)^2$, where x_j^* is her equilibrium action in the unique SPNE of the two stage game. The reason is that j, after observing the actions of seed players, is necessarily playing her best response against other players in $\mathcal{B}$.

However, for any seed agent $i \in \mathcal{A}$, her incentive is slightly different due to the forward-looking feedback from agents in $\mathcal{B}$. To see that, we first note that in the subgame after players $\mathcal{A}$ have chosen their action profile $\mathbf{x}_{\mathcal{A}}$, the unique outcome induced among players in $\mathcal{B}$ is given by

$$\mathbf{x}_{\mathcal{B}}^*(\mathbf{x}_{\mathcal{A}}) = [\mathbf{I} - \delta \mathbf{G}_{\mathcal{B}\mathcal{B}}]^{-1}(\mathbf{a}_B + \delta \mathbf{G}_{BA}\mathbf{x}_A), \tag{24}$$

Given the feedback from $\mathcal{B}$ in the subgame among nodes in B, the nodes in $\mathcal{A}$ choose x_i to maximize:

$$u_i = a_i x_i - \frac{1}{2}x_i^2 + \delta x_i \left(\sum_{i' \in A} g_{ii'} x_{i'} + \sum_{j \in B} g_{ij} x_j(\mathbf{x}_A) \right), \quad \forall i \in \mathcal{A}, \tag{25}$$

where each j's conditional response $x_j^*(\mathbf{x}_A)$ is given by (24). From this reduced form of player i in (25), i's equilibrium payoff must equal

$$\pi_i^*(\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})) = \theta_i(x_i(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}))^2$$

by the previous mathematical observation, where θ_i measures the curvature of i's reduced payoff (see (25)), and equals

$$\theta_i = \frac{1}{2} - \delta^2(\mathbf{G}_{\mathcal{A}\mathcal{B}}[\mathbf{I} - \delta\mathbf{G}_{\mathcal{B}\mathcal{B}}]^{-1}\mathbf{G}_{\mathcal{B}\mathcal{A}})_{ii}.$$

Clearly $\theta_i < 1/2$, implying that

$$\pi_i^*(\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})) = \theta_i(x_i(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}))^2 < \frac{1}{2}(x_i(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}))^2.$$

Nevertheless, we claim that i's payoff is still higher than what she obtains at the state NE. The proof follows from a simple revealed preference argument. Suppose $i \in \mathcal{A}$, instead of choosing $x_i(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$, deviates to her static NE action b_i, while any $i' \in \mathcal{A}$ still choose $x_{i'}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$. To simplify the notation, we define the vector $\hat{\mathbf{x}}_{\mathcal{A}}$ as the action profile among agents in $\mathcal{A}$ under this deviation. Obviously, i's equilibrium payoff at the SPNE $\mathbf{x}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ is higher than i's payoff after the deviation to $\hat{\mathbf{x}}_{\mathcal{A}}$. Moreover, recall

$$x_{i'}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \geq b_{i'}(\mathbf{G}, \delta; \mathbf{a}) \tag{26}$$

by Lemma 2, so $\hat{\mathbf{x}}_{\mathcal{A}} \succeq \mathbf{b}_{\mathcal{A}}$. As $\mathbf{x}^*_{\mathcal{B}}(\cdot)$ increases with $\mathbf{x}_{\mathcal{A}}$ by (24),

$$\mathbf{x}^*_{\mathcal{B}}(\hat{\mathbf{x}}_{\mathcal{A}}) \succeq \mathbf{x}^*_{\mathcal{B}}(\mathbf{b}_{\mathcal{A}}) = \mathbf{b}_{\mathcal{B}} \tag{27}$$

where the last step follows from the static NE.

Therefore, i's payoff after this deviation, by (25), would be

$$a_ib_i - \frac{1}{2}b_i^2 + \delta b_i\left(\sum_{j\in A} g_{ii'}x_{i'}(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) + \sum_{j\in B} g_{ij}x_j^*(\hat{\mathbf{x}}_{\mathcal{A}})\right) \geq a_ib_i - \frac{1}{2}b_i^2 + \delta b_i \sum_k g_{ik}b_k = \frac{1}{2}b_i^2.$$

Here in the first inequality we use (26) and (27), in the last step we use i's equilibrium payoff at static NE (see part (i) of Lemma 5).

□

A.7 Proof of Theorem 4

Given $\mathcal{A}, \mathbf{p}$, the demand profile in the sequential consumption game is given by

$$\mathbf{X}(\mathbf{G}, \delta, \mathcal{A}, \mathbf{p}) = \mathbf{Z}(\delta)(\mathbf{a} - \mathbf{p}),$$

where $\mathbf{Z}$ is given by (4.3), and depends on $\mathcal{A}, \mathbf{G}, \delta$. So, the optimal pricing solves

$$\max_{\mathbf{p}}\langle \mathbf{p} - \mathbf{c}, \mathbf{X}(\mathbf{G}, \delta, \mathcal{A}, \mathbf{p})\rangle = \max_{\mathbf{p}}\langle \mathbf{p} - \mathbf{c}, \mathbf{Z}(\delta)(\mathbf{a} - \mathbf{p})\rangle,$$

and yields

$$\mathbf{p}^* = \frac{\mathbf{a}+\mathbf{c}}{2},$$

and

$$\Pi^*(\mathbf{G}, \delta; \mathcal{A}) = \frac{1}{4}(\mathbf{a}-\mathbf{c})'\mathbf{Z}(\delta)(\mathbf{a}-\mathbf{c}).$$

When $\mathcal{A} = \emptyset$, $\mathbf{Z}(\delta) = \mathbf{M}(\delta)$ and thus $\Pi^*(\mathbf{G}, \delta; \emptyset) = \frac{1}{4}(\mathbf{a}-\mathbf{c})'\mathbf{M}(\delta)(\mathbf{a}-\mathbf{c})$. By Theorem 1, $\mathbf{Z}(\delta) \preceq \mathbf{M}(\sqrt{2}\delta)$ for any $\mathcal{A}$.

Given $\mathbf{c} = \mathbf{0}$ and $\mathbf{a} = \mathbf{1}$, it is easy to see that $\gamma^f = \gamma$. □

Remark 5. *Our proof shows that, without the conditions $\mathbf{c} = \mathbf{0}$ and $\mathbf{a} = \mathbf{1}$, we still have $\gamma^f < \sqrt{2}$.*

A.8 Proof of Proposition 2

(i) From the definitions of γ and γ_ρ, we have

$$b(\mathbf{G}, \gamma\delta; \mathbf{a}) - \rho(\mathcal{A}) = b(\mathbf{G}, \gamma_\rho\delta; \mathbf{a}). \tag{28}$$

Since $\rho(\mathcal{A}) \geq 0$, we have

$$b(\mathbf{G}, \gamma_\rho\delta; \mathbf{a}) \leq b(\mathbf{G}, \gamma\delta; \mathbf{a}),$$

implying that $\gamma_\rho(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a}) \leq \gamma(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$. The latter is less than $\sqrt{2}$ by Theorem 1.

(ii) Consider the Dyad network $\mathbf{G}$ in Example 1 and take $\mathcal{A} = \{1\}$ and $a_1 = a_2 = 1$. From the definition, $\gamma_\rho(\mathbf{G}, \delta, \mathcal{A}; \mathbf{a})$ solves the following equation

$$\frac{2(1+\delta)-\delta^2}{1-2\delta^2} - \rho(\{1\}) = 2\frac{1+\gamma_\rho\delta}{1-(\gamma_\rho)^2\delta^2}. \tag{29}$$

The range of δ is $[0, 1/\sqrt{2})$. Though the exact expression of γ_ρ can be solved using the quadratic form, for our purpose, we only need to show that as δ approaches $1/\sqrt{2}$, γ_ρ approaches $\sqrt{2}$. To see that, we observe that the left-hand side of (29) approaches to infinite as $\delta \to 1/\sqrt{2}$, therefore, there must be $\lim_{\delta\to 1/\sqrt{2}}(1-(\gamma_\rho)^2\delta^2) = 0$. Consequently, $\lim_{\delta\to 1/\sqrt{2}} \gamma_\rho = \sqrt{2}$, which proves part (i).

□

A.9 Proof of Proposition 4

Part (i) follows from Proposition 7 of Zhou and Chen (2015) and definition of γ. For part (ii), we prove it by induction on the depth k of the sequence $\mathcal{S}$.

When $k = 1$, the game has only one stage. For any $\mathbf{G}, \delta, \mathbf{a}$, $\gamma = 1 < \sqrt{2}$.

When $k = 2$, then we are back to the case of Theorem 1, which implies $\gamma \leq \sqrt{2}$ for any $\mathbf{G}$, δ, $\mathcal{A}$ and $\mathbf{a}$.

Suppose $\gamma(\mathbf{G}, \delta, \mathcal{S}'; \mathbf{a}) \leq \sqrt{2}$ for any sequence $\mathcal{S}'$ with depth no greater than $k-1$.

Now consider any sequence $\mathcal{S} = (P_1, P_2, \dots, P_k)$ with depth k.

Given $\mathbf{G}, S, \delta$, recall $\mathbf{x}^S$ is the equilibrium effort in the sequential game. As shown in the proof of Proposition 7 in Zhou and Chen (2015), there exists matrix $\mathbf{Z}(\delta)$ such that $\mathbf{x}^S(\delta) = \mathbf{Z}(\delta)\mathbf{a}$. We want to show $\gamma(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a}) \leq \sqrt{2}$ holds for any sequence with depth k as well.

We first start with a characterization of matrix $\mathbf{Z}$ using induction. Let $\mathcal{A} = P_1$ and $B = \mathcal{N} \backslash \mathcal{A} = \cup_{i=2}^k P_i$. Let $\mathcal{S}' = (P_2, \dots, P_k)$. After players in $\mathcal{A}$ have moved, the equilibrium in the subgame with sequence $\mathcal{S}'$ is given by

$$\mathbf{x}_B = \tilde{\mathbf{U}}\mathbf{a}_B = \tilde{\mathbf{U}}(\mathbf{a}_B + \delta \mathbf{G}_{BA}\mathbf{x}_A), \tag{30}$$

with a matrix $\tilde{\mathbf{U}}$. Such a matrix $\tilde{\mathbf{U}}$ is called feedback matrix in Zhou and Chen (2015) as it determines how responses are followers' actions to the action of the nodes in A. Since the sequence $\mathcal{S}'$ has depth $k-1$, by induction hypothesis, $\tilde{\mathbf{U}} \leq [1 - \sqrt{2}\delta \mathbf{G}_{BB}]^{-1}$.

Given the feedback matrix $\tilde{\mathbf{U}}$ in the subgame among nodes in B, the nodes in $\mathcal{A}$ choose x_i to maximize:

$$u_i = a_i x_i - \frac{1}{2}x_i^2 + \delta x_i \left(\sum_{j \in A} g_{ij}x_j + \sum_{j \in B} g_{ij}x_j(\mathbf{x}_A) \right), \quad \forall i \in \mathcal{A}, \tag{31}$$

where each $x_j(\mathbf{x}_A)$ is given by (30). Solving the subgame among nodes $\mathcal{A}$ using the above payoffs yields the equilibrium outcomes for nodes in A, and in B by (30). Explicitly, We obtain:[31]

$$\mathbf{Z}(\delta) = \begin{pmatrix} \left[\mathbf{I} - \delta(\tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D)\right]^{-1} & \delta \left[\mathbf{I} - \delta(\tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D)\right]^{-1} \mathbf{G}_{AB}\mathbf{U} \\ \delta \tilde{\mathbf{U}}\mathbf{G}_{BA} \left[\mathbf{I} - \delta(\tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D)\right]^{-1} & \tilde{\mathbf{U}} + \delta^2 \tilde{\mathbf{U}}\mathbf{G}_{BA} \left[\mathbf{I} - \delta(\tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D)\right]^{-1} \mathbf{G}_{AB}\mathbf{U} \end{pmatrix}, \tag{32}$$

where $\tilde{\mathbf{U}}$ is the feedback matrix in the subgame $(P_2, \dots, P_k)$, and $\tilde{\mathbf{T}} = \mathbf{G}_{AA} + \delta \mathbf{G}_{AB}\tilde{\mathbf{U}}\mathbf{G}_{BA}$.

For the one-stage game with $\sqrt{2}\delta$, the equilibrium effort is $\mathbf{b}(\sqrt{2}\delta) = [\mathbf{I} - \sqrt{2}\delta\mathbf{G}]^{-1}\mathbf{a}$. Similar to equation (18), the feedback matrix is

$$[\mathbf{I} - \sqrt{2}\delta\mathbf{G}]^{-1} = \begin{pmatrix} \left[\mathbf{I} - \sqrt{2}\delta(\mathbf{T} + 0)\right]^{-1} & \sqrt{2}\delta \left[\mathbf{I} - \sqrt{2}\delta(\mathbf{T} + 0)\right]^{-1} \mathbf{G}_{AB}\mathbf{U} \\ \sqrt{2}\delta \mathbf{U}\mathbf{G}_{BA} \left[\mathbf{I} - \sqrt{2}\delta(\mathbf{T} + 0)\right]^{-1} & \mathbf{U} + 2\delta^2 \mathbf{U}\mathbf{G}_{BA} \left[\mathbf{I} - \delta(\mathbf{T} + 0)\right]^{-1} \mathbf{G}_{AB}\mathbf{U} \end{pmatrix}, \tag{33}$$

where $\mathbf{T} := \mathbf{T}(\sqrt{2}\delta) = \mathbf{G}_{AA} + \sqrt{2}\delta\mathbf{G}_{AB}\,\mathbf{U}\,\mathbf{G}_{BA}$, and $\mathbf{U} := \mathbf{U}(\sqrt{2}\delta) = [\mathbf{I} - \sqrt{2}\delta\mathbf{G}_{BB}]^{-1}$.

We only need to show $\mathbf{Z}(\delta) \precsim [\mathbf{I} - \sqrt{2}\delta\mathbf{G}]^{-1}$, for which it suffices to show

$$\left[\mathbf{I} - \delta(\tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D)\right]^{-1} \precsim \left[\mathbf{I} - \sqrt{2}\delta(\mathbf{T} + 0)\right]^{-1},$$

[31] By the same analysis as the case with two stages in the proof of Proposition 1 in Zhou and Chen (2015), except that the matrix $\tilde{\mathbf{U}}$ is given by the induction step.

which is implied by

$$\mathbf{I} - \sqrt{2}\delta(\mathbf{T} + 0) \precsim \mathbf{I} - \delta(\tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D) \Leftrightarrow \tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D \precsim \sqrt{2}\mathbf{T}.$$

Substituting $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{U}}$ implies

$$\begin{aligned} \tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D &= \mathbf{G}_{AA} + \delta \mathbf{G}_{AB}\tilde{\mathbf{U}}\mathbf{G}_{BA} + \mathbf{G}_{AA}^D + \delta(\mathbf{G}_{AB}\tilde{\mathbf{U}}\mathbf{G}_{BA})^D \\ &\precsim \mathbf{G}_{AA} + 2\delta \mathbf{G}_{AB}[\mathbf{I} - \sqrt{2}\delta \mathbf{G}_{BB}]^{-1}\mathbf{G}_{BA}, \end{aligned}$$

where the $\precsim$ step uses: (i) $\mathbf{G}_{AA}^D = \mathbf{0}$; (ii) the induction hypothesis $\tilde{\mathbf{U}} \precsim [\mathbf{I} - \sqrt{2}\delta \mathbf{G}_{BB}]^{-1}$; (iii) the matrices $\mathbf{G}_{AB}, \mathbf{G}_{BA}$ are positive and $\tilde{\mathbf{U}}$ is also positive for δ sufficiently small, so $(\mathbf{G}_{AB}\tilde{\mathbf{U}}\mathbf{G}_{BA})^D \precsim \mathbf{G}_{AB}\tilde{\mathbf{U}}\mathbf{G}_{BA}$.

Again substituting $\mathbf{T}$ and $\mathbf{U}$ implies

$$\sqrt{2}\mathbf{T} = \sqrt{2}\mathbf{G}_{AA} + 2\delta \mathbf{G}_{AB}[\mathbf{I} - \sqrt{2}\delta \mathbf{G}_{BB}]^{-1}\mathbf{G}_{BA}.$$

Since $\mathbf{G}_{AA} \succsim 0$, the above implies $\sqrt{2}\mathbf{T} \succsim \tilde{\mathbf{T}} + \tilde{\mathbf{T}}^D$. □

A.10 Multi-stage sequences with revisions

A *generalized sequence* $\tilde{\mathcal{S}} = (\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_k)$ is a sequence of subsets of $\mathcal{N}$ such that $\cup_{1 \leq i \leq k} \tilde{P}_i = \mathcal{N}$, where $|\tilde{\mathcal{S}}| = k$ corresponds to the number of steps of this sequence.

Similar to Section 5.4, we can define an extensive-form game with complete information where players move according to the sequence in $\tilde{\mathcal{S}}$. That is, players in $\tilde{P}_t \subseteq \mathcal{N}$ moves simultaneously to choose $(x_{i,t})_{i \in \tilde{P}_t}$ in stage $t = 1, \ldots, k$.

Let $H_t = \prod_{s \leq t-1}((X_{i,s})_{i \in \tilde{P}_s})$ be the set of histories at the beginning of stage $t = 1, \ldots, k$ with $X_{i,s} = \mathbb{R}_+$. A generic history is $h_t \in H_t$. Player i's strategy is $s_i = (s_{i,t})_{i \in \tilde{P}_t, t=1,\ldots,k}$, where $s_{i,t} : H_t \mapsto \mathbb{R}_+$ if $i \in \tilde{P}_t$. Let $\mathbf{s}$ denote the profile of strategies.

For player $i \in \mathcal{N}$, let $\tau(i)$ be the last stage i can move according to $\tilde{\mathcal{S}}$. Define the vector $\mathbf{x} := (x_{1,\tau(1)}, \ldots, x_{n,\tau(n)})$ as the profile of action last taken by each player. Assume only this final action profile $\mathbf{x}$ is payoff relevant for all players; i.e., for all i, $u_i(\cdot)$ is a function of $\mathbf{x}$. Denote by $(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a})$ the k-stage sequential game with overlapping target $\tilde{\mathcal{S}}$. Let $\tilde{\mathbf{s}}$ be a SPNE of this game, and let $\tilde{\mathbf{x}}(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a})$ be the payoff relevant SPNE outcome.

For every generalized sequence $\tilde{\mathcal{S}}$, let $\mathcal{S} = (P_1, P_2, \cdots, P_k)$ be the *reduction of* $\tilde{\mathcal{S}}$ such that $P_t := (\cup_{s \geq t} \tilde{P}_s) \backslash (\cup_{s \geq t+1} \tilde{P}_s)$ for all $t = 1, \ldots, k$ (with $\tilde{P}_{k+1} = \emptyset$). Note that $\mathcal{S}$ is a partition; each cell $P_t \subseteq \tilde{P}_t$ and it contains all the players whose last move is at stage t according to $\tilde{\mathcal{S}}$; i.e., $P_t = \{i \in \mathcal{N} : \tau(i) = t\}$.

We say a generalized k-stage sequential game $(\mathbf{G}, \delta, \mathcal{S}; \cdot)$ is *permissible* if the game $(\mathbf{G}, \delta, \mathcal{S}; \cdot)$ is permissible, with $\mathcal{S}$ being the reduction of $\tilde{\mathcal{S}}$.

Lemma 6. *For all generalized sequence $\tilde{\mathcal{S}}$ and its reduction $\mathcal{S}$, if the game $(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \cdot)$ is permissible, then*

$$\mathbf{x}(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a}) = \tilde{\mathbf{x}}(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a}). \tag{34}$$

Proof. We prove by induction on the number of steps k in $\tilde{\mathcal{S}}$. When $k = 1$, it is the simultaneous-move game. We have $\tilde{\mathcal{S}} = \mathcal{S}$ and eqn (34) holds trivially. When $k = 2$, we have the two-stage game and eqn (34) follows from Lemma 4. Suppose that

$$\mathbf{x}(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a}) = \tilde{\mathbf{x}}(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a})$$

holds for all sequences $\tilde{\mathcal{S}}$ and its reduction $\mathcal{S}$ with less than k steps.

For a general k-step sequence $\tilde{\mathcal{S}} = (\tilde{P}_1, \tilde{P}_2, \cdots, \tilde{P}_k)$. Recall its partitional reduction $\mathcal{S} = (P_1, P_2, \cdots, P_k)$ is given by $P_t = (\cup_{s \geq t} \tilde{P}_s) \backslash (\cup_{s \geq t+1} \tilde{P}_s)$ for all $t = 1, \ldots, k$. Note that $P_t \subseteq \tilde{P}_t$ and $P_t = \{i \in \mathcal{N} : \tau(i) = t\}$ denotes all nodes whose last move is at stage t.

Let $A = P_1$ and $B = \cup_{t=2}^{k} P_t$. At stage 2, the subgame at any history $h_2 = \mathbf{x}_{\tilde{P}_1,1}$ is a general $(k-1)$-step sequential game with sequence $(\tilde{P}_2, \cdots, \tilde{P}_k)$. Still $(P_2, \cdots, P_k)$ is the partitional reduction of $(\tilde{P}_2, \cdots, \tilde{P}_k)$ for players in B. Note that for all nodes in $\tilde{P}_1$, only nodes in $P_1 = A$ have made credible moves at stage 1. Hence, at history $h_2 = \mathbf{x}_{\tilde{P}_1,1}$, only the vector $\mathbf{x}_{A,1}$ is payoff relevant for the continuation subgame.

By induction hypothesis, we have $\mathbf{x}_B(\cdot) = \tilde{\mathbf{x}}_B(\cdot)$ for the corresponding subgame. By the proof of Proposition 7 in Zhou and Chen (2015), $\mathbf{x}_B = U\mathbf{a}'_B = U(\mathbf{a}_B + \delta \mathbf{G}_{BA}\mathbf{x}_{A,1})$, where U is the response matrix. Combining the two observations, we have $\tilde{\mathbf{x}}_B = U(\mathbf{a}_B + \delta \mathbf{G}_{BA}\tilde{\mathbf{x}}_{A,1})$.

Going backward to stage 1, for all nodes $i \in \tilde{P}_1$, we have two cases:

(1) For $i \in \tilde{P}_1 \backslash A$, the node i will have the opportunity to revise his action later at some stage $t > 1$. Hence, it is common knowledge that $x_{i,1}$ is not a credible move and it is payoff-irrelevant. Hence player i is indifferent among all $x_{i,1}$. By the induction step, the best response of players in B, $\tilde{\mathbf{x}}_B(\mathbf{x}_{A,1})$ does not depend on $x_{i,1}$.

(2) For $i \in A$ whose last move in at stage 1, player i chooses $x_{i,1}$ to maximize payoff

$$u_i(\mathbf{x}_{A,1}) = a_i x_i - \frac{1}{2}x_i^2 + \delta x_i \left(\sum_{j \in A} g_{ij} x_{j,1} + \sum_{j \in B} g_{ij} \tilde{x}_j(\mathbf{x}_{A,1}) \right)$$

Note that $\tilde{x}_j(\mathbf{x}_{A,1}) = x_j(\mathbf{x}_{A,1})$ for all follower $j \in B$. Therefore, player $i \in A$'s stage-1 problem in the game $(\mathbf{G}, \delta, \tilde{\mathcal{S}}; \mathbf{a})$ is identical to player $i \in A$'s problem in the game $(\mathbf{G}, \delta, \mathcal{S}; \mathbf{a})$. Hence, the equilibrium actions of the leaders in A are equivalent to those in the reduced game; i.e., $\tilde{\mathbf{x}}_A = \mathbf{x}_A$. In sum, in the game with k-step overlapping sequence, $\tilde{\mathbf{x}}_A = \mathbf{x}_A$ and the response function $\tilde{\mathbf{x}}_B(\tilde{\mathbf{x}}_A) = \mathbf{x}_B(\mathbf{x}_A)$. Thus we prove the claim in eqn (34). □

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B Supplementary Materials

B.1 Heterogeneous marginal utilities

In proving Theorem 1 and Proposition 4, we show a stronger result: the matrix $\mathbf{M}(\sqrt{2}\delta)$ dominates matrix $\mathbf{Z}(\delta)$ entry by entry. As a result, the upper bound $\sqrt{2}$ for the RNSE index γ works in all network games with arbitrary marginal utilities profile $\mathbf{a}$, which does not need to be homogenous across agents. However, when $\mathbf{a}$ is heterogenous, the exact value of the index γ can be affected by the relative intensity of marginal utilities across players. The example below illustrate that, in the leading dyad network, when $a_1 \neq a_2$, the γ index also depends on the relative ratio of a_1/a_2.

Example 1 (continued): *Suppose $a_1 \neq a_2$ and player 1 is selected as the first mover. In this case, the corresponding γ solves the following equation:*

$$\frac{a_1 + \delta a_2}{1 - 2\delta^2} + \frac{\delta a_1 + (1-\delta^2) a_2}{1 - 2\delta^2} = \frac{(a_1 + a_2)}{(1 - \gamma\delta)},$$

where the LHS is the total effort with targeting, while RHS is the total effort without targeting but an adjusted δ. Solving it yields

$$\gamma(a_1, a_2, \delta) := \frac{a_1(1+2\delta) + a_2(1+\delta)}{a_1(1+\delta) + a_2(1+\delta-\delta^2)}.$$

Note that $\gamma(a_1, a_2, \delta)$ increases in δ and a_1 but decreases in a_2.[32]

$$\gamma(0^+, a_2, \delta) = \frac{1+\delta}{1+\delta-\delta^2} \leq \gamma(a_1, a_2, \delta) \leq \gamma(+\infty, a_2, \delta) = \frac{1+2\delta}{1+\delta}.$$

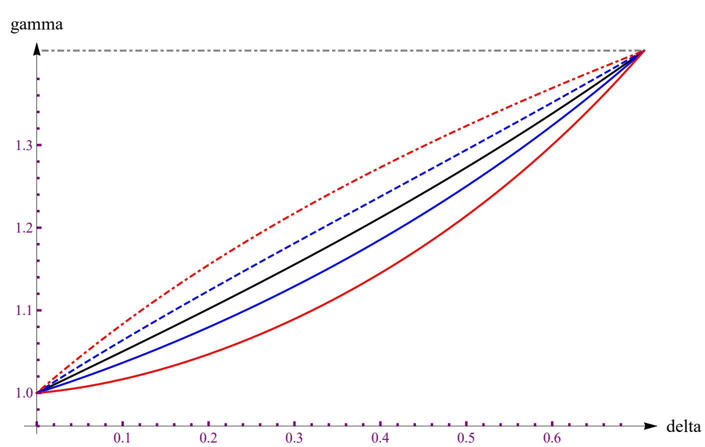

Figure 6: The $\gamma(\cdot)$ curves as functions of δ, for different parameter values of a_1 and a_2. Fix a_2 at 1. In the figure, a_1 equals 10 in the dashed red curve, 2 in the dashed blue curve, 1 in the black curve, 1/2 in the blue curve, and 1/10 in the red curve. The horizontal line is $\gamma = \sqrt{2} \approx 1.414$.

[32] Suppose $a_1 \geq a_2$, then $\gamma(a_1, a_2, \delta) \geq \gamma(a_2, a_1, \delta)$, i.e., targeting node 1 is better than targeting 2.

In Figure 6, all curves are monotone and bounded above by $\sqrt{2}$ and below by 1, for all $\delta \in [0, 1/\sqrt{2})$. So $\sqrt{2}$ remains an upper bound, which is not obtainable exactly, but it is approachable when δ is sufficiently close to $1/\sqrt{2}$.

B.2 γ in specific networks

In this appendix, we will investigate explicit formula of γ and its bounds in specific networks such as complete graphs and bipartite graphs. Leveraging the network structure in $\mathbf{G}$, we can provide upper bound for γ that is still simple but tighter than the universal bound of $\sqrt{2}$ characterized by Theorem 1. To simplify exposition, we will assume players have homogeneous marginal utility from actions, i.e., for all $i \in \mathcal{N}$, $a_i = 1$.

B.2.1 Complete graphs

For a complete graph $\mathbf{K}_n$ with n nodes, the adjacency matrix $\mathbf{G}$ is given by $g_{ii} = 0$ for all i and $g_{ij} = 1$ for all $i \neq j$. Its spectral radius ($\lambda_{\max}(\mathbf{K}_n)$) is $(n-1)$. Zhou and Chen (2015) characterize the equilibria following any targeting policy in complete graphs.

In the simultaneous-move game with synergy $\gamma\delta$, the aggregate action is

$$b(\mathbf{K}_n, \gamma\delta) = \frac{n}{1 - \gamma\delta(n-1)}.$$

In the sequential-move game with synergy δ, by symmetry, the SPNE total action only depends on the number of seeds m the designer chooses to target. Let any policy with m seeds be $\mathcal{A}_m$. In SPNE, the aggregate action is

$$x^S(\mathbf{K}_n, \delta, \mathcal{A}_m) = \frac{n-m}{1-(n-m-1)\delta} + \frac{1+\delta}{1-(n-m-1)\delta} \frac{(1+\delta)m}{1-\delta(n-2)-\delta^2(2n-m-1)}$$

By definition, the index $\gamma(\delta, \mathbf{K}_n; \mathcal{A}_m)$ is given by $x^S(\mathbf{K}_n, \delta; \mathcal{A}_m) = b(\mathbf{K}_n, \gamma\delta)$. Equation (36) provides an explicit formula of $\gamma(\cdot,\cdot,\cdot)$ as a function of m, n, δ, which follows by straightforward calculations.

Lemma 7. *For a complete network $\mathbf{K}_n$, δ is permissible for all targeting policy $\mathcal{A}_m$ if and only if*

$$0 \leq \delta < \bar{\delta}_1 := \frac{2}{n-2+\sqrt{n^2+4n-4}}.$$

Proof. The condition for the simultaneous-move game ($\mathcal{A}_0 = \{\emptyset\}$) to have stable NE is equivalent to $\delta < \frac{1}{\lambda_{\max}(\mathbf{K}_n)} = \frac{1}{n-1}$. And the condition for the sequential-move game with m leaders to have stable SPNE is equivalent to having a quadratic term, $B(\delta, m, n) := 1 - \delta(n-2) - \delta^2(2n-m-1)$ to be non-negative for all $m = 1, 2, \ldots, n$. When the quadratic function equals to 0, the smaller root is negative. The larger root, denoted $\bar{\delta}_m$, is positive and

$$\frac{1}{\bar{\delta}_m} = \frac{(n-2) + \sqrt{(n-2)^2 + 4(2n-m-1)}}{2} \geq (n-2+n)/2 = n-1.$$

Observe that $\bar{\delta}_m < \frac{1}{n-1}$ and it is increasing in m. Hence, the conditions for equilibrium stability is equivalent to

$$0 \leq \delta < \bar{\delta}_1 = \frac{2}{n-2+\sqrt{n^2+4n-4}}.$$

$\square$

Lemma 8. *In a complete graph $\mathbf{K}_n$, the index $\gamma(\mathbf{K}_n, \delta; \mathcal{A}_m)$ is increasing in δ for all permissible $\delta \in (0, \bar{\delta}_1)$.*

Proof. As stated in the main text, in a complete graph with n nodes and $1 \leq m \leq n$ seeds, γ is defined the scaler where the aggregate effort in the sequential game $x^S(\mathbf{K}_n, \delta; \mathcal{A}_m)$ is equal to that in the scaled simultaneous-move game $b(\mathbf{K}_n, \gamma\delta)$, i.e., γ solves the equation

$$\frac{n-m}{1-(n-m-1)\delta} + \frac{1+\delta}{1-(n-m-1)\delta}\frac{(1+\delta)m}{1-\delta(n-2)-\delta^2(2n-m-1)} = \frac{n}{1-\gamma\delta(n-1)}, \tag{35}$$

By straightforward but tedious calculations, γ can be expressed explicitly as a function of δ, m, and n:

$$\gamma := \gamma(\mathbf{K}_n, \delta; \mathcal{A}_m) \quad = \quad \frac{1}{n-1}\frac{n(n-m-1)B + m\delta(2n-m) + mn}{(n-m)B + (1+\delta)^2 m}, \tag{36}$$

where $B := 1 - \delta(n-2) - \delta^2(2n-m-1)$.

Taking partial derivative of equation (36) implies

$$\frac{\partial\gamma}{\partial\delta} = \frac{1}{n-1} \cdot \frac{m(n-m)[n + 2n\delta + (m^2+n^2-2mn+n)\delta^2]}{(-n+(n-m-2)n\delta + (m^2-3mn+n(2n-1))\delta^2)^2}. \tag{37}$$

The sign of equation (37) is determined by the sign of

$$n + 2n\delta + (m^2+n^2-2mn+n)\delta^2 = n + 2n\delta + ((n-m)^2+n)\delta^2.$$

Clearly, for all $\delta \in (0, \bar{\delta}_1)$, all n and $m = 1, \ldots, n$, the above term is positive.

$\square$

Proposition 6. *For a complete graph with n nodes, and for any targeting policy $\mathcal{A}_m \subseteq \mathcal{N}$,*

$$\gamma(\mathbf{K}_n, \delta; \mathcal{A}_m) < \frac{n-2+\sqrt{n^2+4n-4}}{2(n-1)} := \gamma^*(n)$$

for all permissible $\delta \in (0, \bar{\delta}_1)$. Moreover, γ converges to the upper bound $\gamma^*(n)$ as $\delta \to \bar{\delta}_1$.

Proof. For all n, $m = 1, \ldots, n$, and permissible $\delta \in (0, \bar{\delta}_1)$, by Lemma 8, $\gamma(\mathbf{K}_n, \cdot; \mathcal{A}_m)$ is increasing. Therefore,

$$1 = \gamma(\mathbf{K}_n, 0; \mathcal{A}_m) < \gamma(\mathbf{K}_n, \delta; \mathcal{A}_m) < \gamma(\mathbf{K}_n, \bar{\delta}_1; \mathcal{A}_m) = \frac{1}{(n-1)\bar{\delta}_1} = \frac{n-2+\sqrt{n^2+4n-4}}{2(n-1)}.$$

□

Remark 6. *When $n = 2$ and $m = 1$, the complete network is the Dyad network, and the upper bound $\gamma^*(n) = \sqrt{2}$ (see Figure 1). In general, the bound $\gamma^*(n)$ is decreasing in the network size n and converges to 1 as $n \to +\infty$. The convergence follows from* $\lim_{n\to+\infty} \frac{n-2+\sqrt{n^2+4n-4}}{2(n-1)} = 1$.

B.2.2 Bipartite graphs

For a complete bipartite graph $\mathbf{K}_{p,q}$ with $\mathcal{N} = \mathcal{P} \cup \mathcal{Q}$, where $p = |\mathcal{P}|$ and $q = |\mathcal{Q}|$ are the group sizes of two disjoint sets $\mathcal{P}$ and $\mathcal{Q}$, respectively. The adjacency matrix is $\mathbf{G}^{(p,q)} = \begin{bmatrix} \mathbf{0} & \mathbf{J}_{p\times q} \\ \mathbf{J}_{q\times p} & \mathbf{0} \end{bmatrix}$, where $\mathbf{J}_{mn}$ is an m by n matrix of ones. The spectral radius of a complete bipartite graph $\mathbf{K}_{p,q}$ (or $\lambda_{\max}(K_{p,q})$) is $\sqrt{pq}$. Without loss, let $\mathcal{P}$ be the smaller group; i.e., $p \leq q$. This includes the star network as a special case ($p = 1$) and q is the number of spokes. Zhou and Chen (2015) characterize NE in the simultaneous-move game and the SPNE in sequential-move games with target $\mathcal{A}$ for complete bipartite graphs, which enables us to compute the RNSE index $\gamma(\mathbf{K}_{p,q}, \delta; \mathcal{A})$. Moreover, Zhou and Chen (2015) show that (for small δ) in a complete bipartite graph $\mathbf{K}_{p,q}$ the designer's optimal target is $\mathcal{A} = \mathcal{P}$. Denote this target by $\mathcal{A}_p$.

Proposition 7. *For a complete bipartite graph $K_{p,q}$, suppose $p \leq q$ and designer adopts the optimal targeting policy $\mathcal{A}_p = \mathcal{P}$. Then*

$$\gamma(\mathbf{K}_{p,q}, \delta; \mathcal{A}_p) < \sqrt{\frac{(1+p)}{p}} \leq \sqrt{2}.$$

Moreover, Definition 2 holds if $\delta \in \left[0, \frac{1}{\sqrt{(p+1)q}}\right)$.

Proof. In the bipartite graph, let x denote the effort by a player in group $\mathcal{P}$ and y denote the effort by a player in group $\mathcal{Q}$.

In the simultaneous-move game with the scaled synergy parameter $\gamma\delta$, the FOCs are $x = 1 + \gamma\delta q y$, $\quad y = 1 + \gamma\delta p x$. The total effort is

$$TE(\gamma\delta) = px + qy = px + q + \gamma\delta pqx = q + p(1 + \gamma\delta q)x,$$

where

$$x = \frac{1 + \delta q}{1 - \gamma^2\delta^2 pq}.$$

In the sequential game with synergy δ, without loss of generality, suppose $p \le q$. Since $a_i = 1$ for all i, by Zhou and Chen (2015) , it is optimal to have the (weakly) smaller group to move first. So the optimal target is $\mathcal{A}_p = \mathcal{P}$. Then the first-order conditions that determines the SPNE are

$$0 = 1 + \delta q - x^S + \delta^2 x^S (p-1) q + \delta 2 x^S q \quad \text{and} \quad y^S = 1 + \delta p x^S.$$

The total effort is

$$TE^S(\delta) = px^S + qy^S = q + p(1+\delta q)x^S,$$

where

$$x^S = \frac{1+\delta q}{1 - \delta^2 q(p+1)}.$$

By Definition 1, $\gamma(\mathbf{K}_{p,q}, \delta; \mathcal{P})$ is the scalar γ that solves $TE(\gamma\delta) = TE^S(\delta)$, i.e.,

$$\frac{(1+\gamma\delta q)^2}{1-\gamma^2\delta^2 pq} = \frac{(1+\delta q)^2}{1-\delta^2 q(p+1)}. \tag{38}$$

Treating the LHS of (38) as a function of the unknown variable $\gamma \ge 1$ and fixed parameters $p > 0, q > 0$, and $\delta > 0$, clearly this function is strictly increasing in γ. When γ takes value $\sqrt{\frac{(1+p)}{p}}$, the LHS of (38) becomes

$$\frac{(1+\sqrt{\frac{(1+p)}{p}}\delta q)^2}{1-\delta^2 q(p+1)},$$

which is clearly greater than the RHS of (38). Hence, the γ that solves equation (38) must satisfy $\gamma < \sqrt{\frac{(1+p)}{p}}$.

Finally, permissibility requires $\gamma\delta < \frac{1}{\sqrt{pq}}$. Using the upper bound of γ, a (weak) sufficient condition for permissibility is

$$0 \le \delta < \frac{1}{\sqrt{pq}} \frac{1}{\sqrt{\frac{(1+p)}{p}}} = \frac{1}{\sqrt{(p+1)q}}.$$

□

Remark 7.

(i) If $p = q = 1$, then $K_{1,1}$ is equal to the dyad network and the upper bound is $\sqrt{2}$ (see Figure 1). And the upper bound $\sqrt{\frac{(1+p)}{p}} < \sqrt{2}$ for larger networks where $p \ge 2$. As $p \to \infty$, $\sqrt{\frac{(1+p)}{p}}$ converges to 1. It suggests that (two-stage) sequential targeting can only have a very small effect in a bipartite network when both groups are large.

(ii) Suppose $\mathcal{P}$ and $\mathcal{Q}$ have the same size; i.e., $p = q$. Then γ has an explicit formula (by

substituting $q = p$ in equation (38)):

$$\gamma(\mathbf{K}_{p,p}, \delta; \mathcal{A}_p) = \frac{2\delta p + \delta + 2}{2\delta p - \delta^2 p + 2},$$

where permissible δ lies in $\left[0, \frac{1}{\sqrt{(p+1)p}}\right)$.

For any p, $\gamma(\cdot)$ is strictly increasing in δ.[33] *Furthermore, $\gamma(\mathbf{K}_{p,p}, \delta; \mathcal{A}_p) \approx 1 + \frac{\delta}{2}$ by Lemma 3. Therefore, for very small δ the value of γ barely vary with the network size p.*

[33] By standard calculations,

$$\frac{\partial\gamma}{\partial\delta} = \frac{(n-p)p(n + 2\delta n + \delta^2((n-p)^2 + n))}{(n-1)(\delta(1+2\delta)n^2 + \delta^2 p^2 - n(1 + \delta(2+p) + \delta^2(1+3p)))^2} > 0.$$