Solution Midterm I

Econ 500, Fall 2009

Q1 (a) Profit function $= P(Q).Q - C(Q)$

$= 14Q - 0.01Q^2 - 4500.$

$$\frac{d\pi}{dQ} = 14 - 0.02Q = 0 \Rightarrow Q^* = 700.$$

To ensure that Q^* is a maximum point, check for $\boxed{\text{Concavity}}$ of the profit function. (You can also alternatively use the second order derivative test for a local maximum.)

$$\frac{d^2\pi}{dQ^2} = -0.02 < 0. \text{ Hence } Q^* \text{ is a maxima.}$$

Maximum profit $= 14.700 - (0.01).(700)^2 - 4500 = 400.$

(b) Price elasticity $= \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{1}{dP/dQ} \cdot \frac{P}{Q}$ (since P is invertible)

$$= 1 - \frac{3000}{Q}$$

$\bar{Q}$ at which elasticity equals (-1), $1 - \frac{3000}{Q} = -1$ or $Q = 1500$

(c) Marginal revenue is given by the first order derivative of, $P(Q).Q$; the total revenue.

$MR = 18 - 0.012Q$

$= 0$ at $Q = 1500.$

(* This is an important result, true for all <u>linear</u> demand functions).

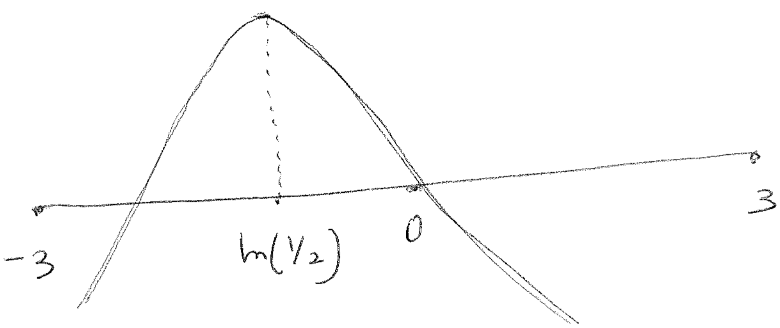
Q2 (a) $f'(x) = -2x + 1 - e^{-x}$

$f''(x) = -2 + e^{-x}$.

For $f'(x)$ to be increasing $f''(x) \geq 0 \Rightarrow x \leq \ln(1/2)$, a negative number.

$f'(x)$ to be decreasing $f''(x) \leq 0 \Rightarrow x \geq \ln(1/2)$

(b)



At $x = \ln(1/2)$, $f'(x) = 2\ln 2 - 1 > 0$, [Note $e = 2.718$ approximately]

$f'(-3) = 7 - e^{3} < 0$

$f'(3) = -5 - e^{-3} < 0$.

Since $f'(x)$ is ~~decreasing~~ increasing over $(-3, \ln(1/2))$ and decreasing over $(\ln(1/2), 3)$, it has $\boxed{a}$ only one root (a zero) over $(-3, \ln(1/2))$ and $\boxed{a}$ only one root over $(\ln(1/2), 3)$.

Hence the function has $\boxed{two}$ stationary points over $[-3, 3]$.

(c) Check that $f'(0) = 0$, hence $x = 0$ is one of the two stationary points in $[-3, 3]$.

The other stationary point in $(-3, \ln(1/2))$ cannot be a maximum, as, $f'(x)$ is positive for x greater than this value.

Hence candidate points for a maximum are $x = 0,\ x = -3,\ x = 3,$

$f(0) = 1, \quad f(-3) = -8 + e^{3} > 1, \quad f(3) = -8 + e^{-3} < f(-3).$

Hence $x = -3$ is the maximum point.

Q3 (a) $f(x, y) = x^3 + y^3 - 3xy$

$$f'_y = 3y^2 - 3x = 0 \Rightarrow y^2 = x \quad \text{—(1)}$$

$$f'_x = 3x^2 - 3y = 0 \Rightarrow x^2 = y. \quad \text{—(2)}.$$

~~(1) and~~ (2) implies, $x^4 = y^2$, Substituting into (1), $x(x^3 - 1) = 0$,

$\Rightarrow$ either $x = 0$ or $x = 1$.

Hence the stationary points are $(0, 0)$ and $(1, 1)$

At $(0, 0)$, $\quad f''_{xx} = 6x = 0, \quad f''_{xy} = -3$

$\quad f''_{yy} = 6y = 0, \quad f''_{xx} \cdot f''_{yy} - (f''_{xy})^2 = -9 < 0$

Hence $(0, 0)$ is a saddle point.

At $(1, 1)$. $\quad f''_{xx} = 6x = 6, \quad f''_{xy} = -3,$

$\quad f''_{yy} = 6y = 6, \quad f''_{xx} f''_{yy} - (f''_{xy})^2 = 36 - 9 = 27 > 0$

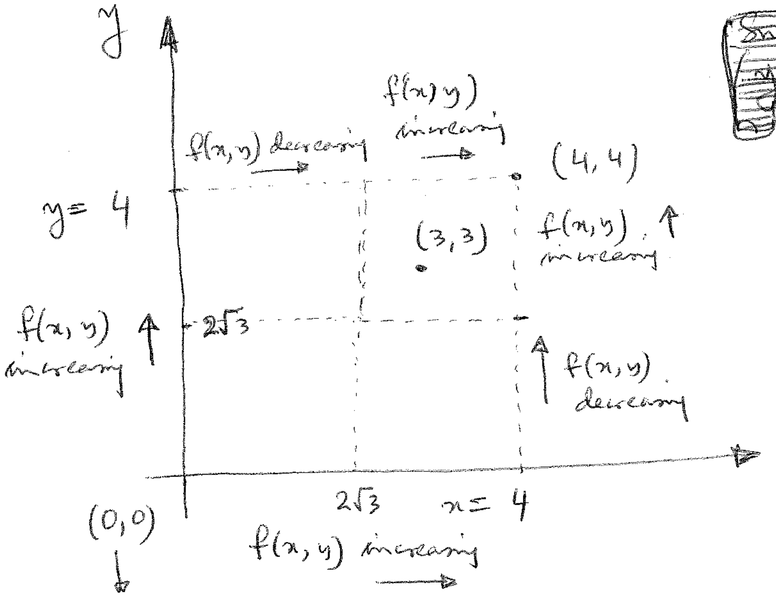
Hence $(1, 1)$ is a strict local minima.

3b By the Weierstrass' Theorem (or the Extreme Value Theorem), the function $f(x,y) = x^3 + y^3 - 9xy + 27$ over the set $\{0 \le x \le 4,\ 0 \le y \le 4\}$ has both a maximum and a minimum, since the function is continuous and the domain is compact.

The extreme points are either interior stationary points or boundary points.

$\boxed{\text{Stationary points of } f(x,y)}$: $\begin{cases} f'_x = 3x^2 - 9y = 0 \Rightarrow x^2 = 3y \\ f'_y = 3y^2 - 9x = 0 \Rightarrow y^2 = 3x. \end{cases}$

Substituting one into the other, and simplifying, the stationary points are $(0,0)$ and $(3,3)$.



$f(0,0) = 27$

Let $y = 4$, For $x \in [0, 2\sqrt{3}]$
$f'_x \le 0 \Rightarrow f(x,y)$ is decreasing in x over this interval

For $x \in [2\sqrt{3}, 4]$, $f'_x \ge 0$
$\Rightarrow f(x,y)$ is increasing in x over this interval.

~~Suppose $y = 0$. Over the interval $[0, 4]$~~

Let $y = 0$, For $x \in [0, 4]$,
$f'_x \ge 0 \Rightarrow f(x,y)$ is increasing in x over this interval

Let $x = 0$, For $y \in [0, 4]$
$f'_y \ge 0 \Rightarrow f(x,y)$ is increasing in y over this interval.

Let $x = 4$, For $y \in [0, 2\sqrt{3}]$
$f'_y \le 0 \Rightarrow f(x,y)$ is decreasing in y over this interval

For $y \in [2\sqrt{3}, 4]$, $f'_y \ge 0$
$\Rightarrow f(x,y)$ is increasing in y over this interval.

Hence Candidate points for maximum and minimum are,

$(0,0)$, $(3,3)$, $(4,4)$, $(4, 2\sqrt{3})$, $(2\sqrt{3}, 4)$, $(0,4)$ and $(4,0)$

$f(0,0) = 27$, $f(3,3) = 0$, $f(4,4) = 11$, $f(4, 2\sqrt{3}) = 91 - 48\sqrt{3} > 0$
$f(0,4) = f(4,0) = 91$ $\qquad = f(2\sqrt{3}, 4)$

Hence $(0,4)$ and $(4,0)$ are maxima and $(3,3)$ is minima.

Q. 4

$$\begin{cases} 2x - 3y + z = 0 \\ x + y - z = 0. \end{cases}$$

The augmented matrix $= \begin{bmatrix} 2, & -3, & 1, & 0 \\ 1, & 1, & -1, & 0 \end{bmatrix} = (A, b)$

$$\simeq \begin{bmatrix} 1, & -3/2, & 1/2, & 0 \\ 1, & 1, & -1, & 0 \end{bmatrix} \simeq \begin{bmatrix} 1, & -3/2, & 1/2, & 0 \\ 0, & 5/2, & -3/2, & 0 \end{bmatrix}$$

$$\simeq \begin{bmatrix} 1, & -3/2, & 1/2, & 0 \\ 0, & 1, & -3/5, & 0 \end{bmatrix} \simeq \begin{bmatrix} 1, & 0, & -2/5, & 0 \\ 0, & 1, & -3/5, & 0 \end{bmatrix}$$

Hence solutions are $\left.\begin{aligned} x &= 2/5\, z \\ y &= 3/5\, z \end{aligned}\right\}$ where z is a free variable.

(*) Note, elementary row operations can be performed in different sequences. Also it is $\boxed{\text{OK}}$ to stop the operations on the matrix, once you have a reduced form system. However for larger systems, it helps to continue as long as possible.

(5) $$\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = \det A$$

Adding rows (2) and (3) to row (1),

$$\det A = \begin{vmatrix} 3a+b, & 3a+b, & 3a+b \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = (3a+b)\begin{vmatrix} 1 & 1 & 1 \\ a & a+b & a \\ a & a & a+b \end{vmatrix}$$

Subtracting column (1) from column (2) and (3),

$$\det A = (3a+b)\begin{vmatrix} 1 & 0 & 0 \\ a & b & 0 \\ a & 0 & b \end{vmatrix} = (3a+b) \cdot b^2.$$

(*) Note, once again, that the same answer can be derived by following different sequences of elementary row or column operations.