HW 3. Total : 40 pts.

13.2 (6 pts)

4. (a) $\pi(x,y) = p\cdot x + q\cdot y - c(x,y)$

$= (25-x)\cdot x + (24-2y)\cdot y - (3x^2+3xy+y^2)$

$= 25x - x^2 + 24y - 2y^2 - 3x^2 - 3xy - y^2$

$= -4x^2 - 3y^2 - 3xy + 25x + 24y$ (1 pt)

(b) $\pi'_x = -8x - 3y + 25 = 0$ (1) (1 pt)

$\pi'_y = -6y - 3x + 24 = 0$ (2) (1 pt)

(1) $\Rightarrow$ $6y + 16x - 50 = 0$ (3)

(2) + (3) $\Rightarrow$ $13x = 26 \Rightarrow x = 2$

substitute into (1) $\Rightarrow -16 - 3y + 25 = 0 \Rightarrow y = 3$ (1 pt)

$\pi''_{xx} = -8 \leq 0$

$\pi''_{yy} = -6 \leq 0$

$\pi''_{xy} = -3$ (2 pts)

$\pi''_{xx}\pi''_{yy} - (\pi''_{xy})^2 = (-8)(-6) - (-3)^2 = 48 - 9 = 39 \geq 0$

Therefore, $x = 2$, $y = 3$ are the values that max. profit.

13.3. 9 pts

6. (a) $Z = -x^4 - y^4$

$Z_x' = -4x^3 \qquad Z_y' = -4y^3$

$Z_{xx}'' = -12x^2 \qquad Z_{yy}'' = -12y^2 \qquad Z_{xy}'' = 0$

At $(x, y) = (0, 0)$, we have $Z_x' = 0$, $Z_y' = 0$

and $AC - B^2 = (-12x^2)(-12y^2) - 0^2 = 0$ (2 pts)

(b) $Z = x^4 + y^4$

$Z_x' = 4x^3 \qquad Z_y' = 4y^3$

$Z_{xx}'' = 12x^2 \qquad Z_{yy}'' = 12y^2 \qquad Z_{xy}'' = 0$

At $(x, y) = (0, 0)$, we have $Z_x' = 0$, $Z_y' = 0$ and $AC - B^2 = 0$ (2 pts)

(c) $Z = x^3 + y^3$

$Z_x' = 3x^2 \qquad Z_y' = 3y^2$

$Z_{xx}'' = 6x \qquad Z_{yy}'' = 6y \qquad Z_{xy}'' = 0$

At $(0, 0)$. $Z_x' = 0$. $Z_y' = 0$ $AC - B^2 = 0$ (2 pts)

on the entire domain.

For (a), $Z = -(x^4 + y^4) \leq 0$; at $(0, 0)$, $Z = 0$

therefore $(0, 0)$ is a maxima. (1 pt)

on the entire domain.

For (b), $Z = x^4 + y^4 \geq 0$; at $(0, 0)$, $Z = 0$

thus $(0, 0)$ is a minima. (1 pt)

For (c). Z takes positive and neg. values at pts arbitrarily close to the origin, thus $(0, 0)$ is a saddle point. (1 pt)

6 pts

13.4. 2 (a) $\pi = p \cdot Q_A + q \cdot Q_B - C(Q_A, Q_B)$

$$= p(a-bp) + q(c-dq) - \alpha - \beta(a+c-bp-dq)$$

$$= -bp^2 - dq^2 + (a+b\beta)p + (c+d\beta)q - \alpha - \beta(a+c)$$

(1 pt)

$$\pi'_p\big|_{p^*} = -2bp + (a+b\beta)\big|_{p^*} = 0$$
$$\pi'_q\big|_{q^*} = -2dq + (c+d\beta)\big|_{q^*} = 0$$
$$\Rightarrow \begin{cases} p^* = \dfrac{a+b\beta}{2b} \\ q^* = \dfrac{c+d\beta}{2d} \end{cases} \to (1\text{ pt})$$

$\pi''_{pp} = -2b \leq 0 \qquad \pi''_{qq} = -2d \leq 0 \qquad \pi''_{pq} = 0$

$\pi''_{pp}\pi''_{qq} - (\pi''_{pq})^2 = 4bd \geq 0$, so (p^*, q^*) max. profits.

(b) Now $p = q = \hat{p}$, so $\pi = -(b+d)\hat{p}^2 + [a+c+(b+d)\beta]\hat{p} - \alpha - \beta(a+c)$

(1 pt)

$$\pi'_{\hat{p}} = -2(b+d)\hat{p} + [a+c+(b+d)\beta] = 0$$

$$\hat{p} = \frac{a+c+(b+d)\beta}{2(b+d)} \to (1\text{ pt})$$

And $\pi''_{\hat{p}\hat{p}} = -2(b+d) \leq 0$, so $\hat{p}$ max. profits.

(c) If $\beta = 0$, then $p^* = \dfrac{a+b\cdot 0}{2b} = \dfrac{a}{2b} \qquad q^* = \dfrac{c}{2d}$

$$\hat{p} = \frac{a+c}{2(b+d)}$$

$$\pi(p^*, q^*) = -b\cdot\left(\frac{a}{2b}\right)^2 - d\left(\frac{c}{2d}\right)^2 + (a+b\beta)\cdot\frac{a}{2b} + (c+d\beta)\cdot\frac{c}{2d}$$
$$- \alpha - \beta(a+c) = \frac{a^2}{4b} + \frac{c^2}{4d} - \alpha \quad (0.5\text{ pt})$$

$$\pi(\hat{p}) = -(b+d)\left[\frac{a+c}{2(b+d)}\right]^2 + (a+c)\cdot\frac{a+c}{2(b+d)} - \alpha$$

$$= \frac{(a+c)^2}{4(b+d)} - \alpha \qquad (0.5 \text{ pt})$$

$$\pi(p^*, q^*) - \pi(\hat{p}) = \frac{-bd(a+c)^2 + (da^2+bc^2)(b+d) \quad + b^2c^2 + d^2a^2 - 2bdac}{4bd(b+d) \qquad\qquad 4bd(b+d)}$$

$$= \frac{(bc-ad)^2}{4bd(b+d)} \geq 0 \qquad (1 \text{ pt})$$

Comment: If $bc = ad$, then $\pi(p^*, q^*) = \pi(\hat{p})$.

And $p^* = q^*$. in this case the firm will charge the same price in each market anyway. (0 pt)

13.5 (10 pts)

4. (a) $f'_x = 2axy + by + 2y^2$

$f'_y = ax^2 + bx + 4xy$

For $(2/3, 1/3)$ to be a local min.

$$\left.\begin{aligned} f'_x(2/3, 1/3) &= 2a\frac{2}{9} + \frac{1}{3}b + \frac{2}{9} = 0 \\ f'_y(2/3, 1/3) &= a\cdot\frac{4}{9} + \frac{2}{3}b + \frac{8}{9} = 0 \end{aligned}\right\} \Rightarrow a = 1,\ b = -2 \quad (1 \text{ pt})$$

And $f(2/3, 1/3) = a\frac{4}{9}\cdot\frac{1}{3} + b\cdot\frac{2}{9} + 2\cdot\frac{2}{3}\cdot\frac{1}{9} + c$

$$= \frac{4}{27} - \frac{4}{9} + \frac{4}{27} + c = \frac{-4}{27} + c = -\frac{1}{9}$$

$$\Rightarrow c = \frac{1}{27} \qquad (1 \text{ pt})$$

Because $A = f''_{xx}(2/3,\ 1/3) = 2\cdot\frac{1}{3} = \frac{2}{3}$

$C = f''_{yy}(2/3,\ 1/3) = \frac{8}{3}$ (1 pt)

$B = f''_{xy}(2/3,\ 1/3) = \frac{4}{3} - 2 + \frac{4}{3} = \frac{2}{3}$

$A > 0$, $AC - B^2 = 4 > 0$, it confirms $(2/3, 1/3)$ is a local min.

(b) $f(x,y) = x^2y - 2xy + 2xy^2 + \frac{1}{27}$ $\quad x \geq 0,\ y \geq 0,\ 2x+y \leq 4$

$f'_x = 2xy - 2y + 2y^2 = 0 \quad (1)$ (1 pt)

$f'_y = x^2 - 2x + 4xy = 0 \quad (2)$

$(1)\times 2 - (2)$: $4y^2 - x^2 + 2x - 4y = 0$

$\Rightarrow (2y-1)^2 - (x-1)^2 = 0 \Rightarrow \begin{cases} 2y-1 = x-1 \Rightarrow x = 2y \\ 2y-1 = -(x-1) \Rightarrow 2y = 2-x \Rightarrow x = 2-2y \end{cases}$

(i) Substitute $x = 2y$ into (1): $6y^2 - 2y = 0 \Rightarrow y(3y-1) = 0$

$y = 0,\ 1/3$, so $x = 0,\ 2/3$

$(0,0)$ is a boundary pt. so the only interior stationary pt is $(2/3, 1/3)$ (1 pt)

(ii) substitute $x = 2-2y$ into (1): $2(2-2y)y - 2y + 2y^2 = 0$

$2y - 2y^2 = 0 \quad y(1-y) = 0 \Rightarrow y = 0, 1$, so $x = 2, 0$

$(2,0)$ and $(0,1)$ are both boundary pt. (1 pt)

Now. check the points on boundary line $2x+y=4$,

i.e. $y = 4-2x$

$$f(x) = x^2(4-2x) - 2x(4-2x) + 2x(4-2x)^2 + \frac{1}{27}$$ (1 pt)

$$f'_x = 2x(4-2x) - 2x^2 - 2(4-2x) + 4x + 2(4-2x)^2 + 4x(4-2x)\cdot(-2)$$

$$= 8x - 4x^2 - 2x^2 - 8 + 4x + 4x + 32 + 8x^2 - 32x - 32x + 16x^2$$

$$= 18x^2 - 48x + 24 = 0 \Rightarrow 3x^2 - 8x + 4 = 0$$

$$\Rightarrow (x-2)(3x-2) = 0, \quad \text{so } x = 2 \text{ or } x = 2/3$$

$$\text{and } y = 0 \text{ or } y = 8/3$$ (1 pt)

Now we have optimality candidates:

$(0, 0)$, $(2/3, 1/3)$, $(0, 1)$, $(2, 0)$, $(2/3, 8/3)$

$f(0,0) = 1/27$ $\quad f(2/3, 1/3) = -1/9$ $\quad f(0,1) = 1/27$

$f(2,0) = 1/27$ $\quad f(2/3, 8/3) = \frac{4}{9}\cdot\frac{8}{3} - 2\cdot\frac{2}{3}\cdot\frac{8}{3} + 2\cdot\frac{2}{3}\cdot\frac{64}{9} + \frac{1}{27} = \frac{193}{27}$

Therefore f has a max. value of $193/27$ at $(2/3, 8/3)$ (1 pt)
and a min value of $-1/9$ at $(2/3, 1/3)$ (1 pt)

7. (3 pts) Let $g(x) = \begin{cases} 1 & \text{if } x < 1 \\ 2 & \text{if } x \geq 1 \end{cases}$

then the set $\{x : g(x) \leq 1\} = (-\infty, 1)$, which is not closed.

Note: other examples are possible.

13.7 (6 pts)

2. $$\left.\begin{aligned} \pi'_K &= \tfrac{2}{3} p K^{-1/3} - r = 0 \\ \pi'_L &= \tfrac{1}{2} p L^{-1/2} - w = 0 \\ \pi'_T &= \tfrac{1}{3} p T^{-2/3} - q = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} K^* &= \tfrac{8}{27} p^3 r^{-3} \\ L^* &= \tfrac{1}{4} p^2 w^{-2} \\ T^* &= \tfrac{1}{3\sqrt{3}} p^{\frac{3}{2}} q^{-\frac{3}{2}} \end{aligned}$$ → 3 pts

$$Q^* = (K^*)^{2/3} + (L^*)^{1/2} + (T^*)^{1/3}$$

$$= \frac{4}{9} p^2 r^{-2} + \frac{1}{2} p w^{-1} + \frac{1}{\sqrt{3}} p^{1/2} q^{-1/2} \longrightarrow 2 \text{ pts}$$

$$\left.\begin{array}{l} \partial Q^* / \partial r = -\frac{8}{9} p^2 r^{-3} \\ \partial K^* / \partial p = \frac{8}{9} p^2 r^{-3} \end{array}\right\} \Rightarrow \frac{\partial Q^*}{\partial r} = -\frac{\partial K^*}{\partial p} \longrightarrow 1 \text{ pt}$$