

The Equivalence between Four Economic Theorems and Brouwer's Fixed Point Theorem*

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Summary: This note provides the equivalence between Brouwer's (or Kakutani's) fixed point theorem and four economic theorems (the existence theorems for competitive equilibrium, Nash equilibrium, core, and hybrid equilibrium). Such equivalence leaves with us three classes of open problems to challenge our mathematical curiosity, one of which is to derive Kakutani's fixed point theorem directly from Nash's theorem (1951).

Keywords: Fixed point theorem, competitive equilibrium, core, Nash equilibrium, hybrid equilibrium.

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This note shows that Brouwer's or Kakutani's fixed point theorem is equivalent to four fundamental theorems in economics and game theory: the existence theorems for the competitive equilibrium [1], the Nash equilibrium [5], the core [6], and the hybrid equilibrium [10]. Since these four economic theorems describe four different types of

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social institutions, their equivalence implies that the outcomes in each of the four institutions can be theoretically implemented by any of the other three institutions. Section 1 below states the equivalence problems, Section 2 states and proves the equivalence theorem, and Section 3 provides a list of open problems for future research.

1. The equivalence problems

Let $N = \{1, \dots, n\}$ be a finite set, a subset $S \neq \emptyset \subset N$ will be called a coalition of players. Given $S \subset N$, let \mathbf{R}^S denote the Euclidean space whose dimension is the number of players in S and whose coordinates are the players in S . Let $\mathbf{e} \in \mathbf{R}^n$ be the vector of ones, $\mathbf{0}$ be the vector of zeros, $\text{Arg-Max}\{f(z) \mid z \in Z\}$ be the optimal set for $\text{Max}\{f(z) \mid z \in Z\}$; for any $x, y \in \mathbf{R}^n$, $x \geq y \Leftrightarrow x_i \geq y_i$, all i ; $x > y \Leftrightarrow x \geq y, x \neq y$; $x \gg y \Leftrightarrow x_i > y_i$, all i ; x_S be the projection of x on \mathbf{R}^S , and (x_S, y_{-S}) be a column vector $z \in \mathbf{R}^n$ such that $z_i = x_i$ if $i \in S$, $z_i = y_i$ if $i \notin S$.

For simplicity of exposition, we will state the next six theorems in their simplest forms. First, Brouwer's and Kakutani's fixed point theorems are given below:

(i) (Brouwer, [2]) Let $X \neq \emptyset \subset \mathbf{R}^n$ be compact and convex, and $f: X \rightarrow X$ be a continuous function. Then there exists at least one $x \in X$ such that $f(x) = x$.

(ii) (Kakutani, [3]) Let $X \neq \emptyset \subset \mathbf{R}^n$ be compact and convex, $f: X \rightarrow 2^X$ be a correspondence. Assume: (a) f is upper semicontinuous; (b) for each $x \in X$, $f(x) \neq \emptyset \subset X$ is closed and convex. Then there exists at least one $x \in X$ such that $x \in f(x)$.

Second, consider the Nash equilibrium for a normal form game: $\Gamma = \{N, X_i, u_i\}$, where for each $i \in N$, $X_i \neq \emptyset \subset \mathbf{R}^{m(i)}$ and $u_i: X = \prod_k X_k \rightarrow \mathbf{R}$ are player i 's choice set and payoff function. A choice $x \in X$ is a **Nash equilibrium** if $x_i \in \text{Arg-Max}\{u_i(y_i, x_{-i}) \mid y_i \in X_i\}$ for each i (i.e., if each x_i is player i 's best response to the complementary choice x_{-i}).

(iii) (Nash, [5]) For each $i \in N$, assume: (a) X_i is compact and convex; (b) $u_i(x)$ is continuous in x and quasi-concave in x_i . Then Γ has at least one Nash equilibrium.

Next, consider the competitive equilibrium in an exchange economy $E = \{N, \omega_i, U_i\}$, where for each $i \in N$, $\omega_i = \{\omega_i^1, \dots, \omega_i^L\}' \in \mathbf{R}_+^L$ and $U_i: \mathbf{R}_+^L \rightarrow \mathbf{R}$ are trader i 's endowments and utility function. Let $p \in \Delta^L = \{p \in \mathbf{R}_+^L \mid \sum_{i=1}^L p^i = 1\}$ be a price vector, $B_i(p) = \{x_i \in \mathbf{R}_+^L \mid px_i \leq p\omega_i = \sum_{k=1}^L p^k \omega_i^k\}$ and $D_i(p) = \text{Arg-Max} \{U_i(x_i) \mid x_i \in B_i(p)\}$ be a trader i 's budget set and demand correspondence. A pair (p, x) is a **competitive equilibrium** if each $x_i \in D_i(p)$ and $\sum x_i \leq \sum \omega_i$, where $x = \{x_1, \dots, x_n\} \in \mathbf{R}_+^{n \times L}$. Adam Smith, the acclaimed first economist, had perceived the existence of such an equilibrium in *Wealth of Nations* (1776): economic activities and resources will be directed by an *invisible hand* to their maximal uses.

(iv) (Arrow and Debreu, [1]) Assume: (a) $\sum \omega_i \gg 0$; (b) for each $i \in N$, $U_i(x_i)$ is continuous, quasiconcave and increasing. Then E has at least one competitive equilibrium.

Now, consider the core for a coalitional NTU (non-transferable utility) game $\Gamma_{CF} = \{N, v(\cdot)\}$, where for each S , $v(S) \neq \emptyset \subset \mathbf{R}^S$ is its set of payoffs satisfying: (a) it is closed and bounded from above; (b) $y \in v(S)$ and $u \leq y$ imply $u \in v(S)$; and (c) $\exists y \in v(S)$ such that $y_i \geq \text{Max} \{x_i \mid x_i \in v(i)\}$, all $i \in S$. Given a set of coalitions $B = \{T_1, \dots, T_k\}$, let $B(i) = \{T \in B \mid i \in T\}$ be the subset of coalitions that include i . B is balanced if $\exists w_T > 0$ for each $T \in B$ such that $\sum_{T \in B(i)} w_T = 1$ holds for all i . A game is balanced if for any balanced B , $u \in v(N)$ must hold if $u_S \in v(S)$ for all $S \in B$. Let $\partial v(S) = \{y \in v(S) \mid \exists \text{ no } x \in v(S) \text{ with } x \gg y\} \neq \emptyset$ be the weakly efficient set of $v(S)$. A vector u is blocked by S if $u_S \in v(S) \setminus \partial v(S)$, $u \in \partial v(N)$ is in the core if it is unblocked by all $S \neq N$. Let $C(\Gamma_{CF}) = \{u \in \partial v(N) \mid u_S \notin v(S) \setminus \partial v(S), \text{ all } S \neq N\}$ denote the core.

(v) (Scarf, [6]) If Γ_{CF} is balanced, then $C(\Gamma_{CF}) \neq \emptyset$.

Finally, consider the hybrid equilibrium for $\Gamma = \{N, X_i, u_i\}$. Let the weakly efficient solution set be $X_{we}^* = \{x \in X \mid \nexists \text{ no } y \in X \text{ with } u(y) \gg u(x)\}$, where $u(x) = \{u_i(x) \mid i \in N\}$, and let the coalitional game derived from Γ in the α -fashion, $\Gamma_\alpha = \{N, v_\alpha(\cdot)\}$, be given by:

$$v_\alpha(S) = \bigcup_{x_S, y_{-S}} \bigcap \{w_S \in \mathbf{R}^S \mid w_S \leq u_S(x_S, y_{-S})\} \text{ for } S \neq N, \text{ and}$$

$$v(N) = v_\alpha(N) = \{w \in \mathbf{R}^n \mid w \leq u(x) \text{ for some } x \in X_{we}^*\}.$$

Then, $x \in X_{we}^*$ is an α -core solution if $u(x) \in C(\Gamma_\alpha)$. A partition $\Delta = \{S_l, \dots, S_k\}$ (i.e., $\cup S_i = N$, $S_i \cap S_j = \emptyset$, all $i \neq j$) defines k parametric games: $\Gamma_S(y_{-S}) = \{S, X_i, u_i(x_S, y_{-S})\}$, $S \in \Delta$. A choice $x = \{x_S \mid S \in \Delta\} \in X$ is a hybrid equilibrium for Δ if each x_S is an α -core solution of $\Gamma_S(x_{-S})$.

(vi) (Zhao, [10]) Given $\Delta = \{S_l, \dots, S_k\}$ in Γ . For each $S \in \Delta$, assume: (a) X_i , $i \in S$, all are compact and convex; (b) $u_i(x) = u_i(x_S, x_{-S})$, $i \in S$, all are continuous in x and quasi-concave in x_S . Then there exists at least one hybrid equilibrium for Δ .

(Figure 1 about here)

As shown in Figure 1, the hybrid equilibria includes the core¹ (i.e., $x(\Delta_m)$ for $\Delta_m = \{N\}$) and the Nash equilibrium (i.e., $x(\Delta_0)$ for $\Delta_0 = \{\{1\}, \dots, \{n\}\}$) as two polar cases.

2. The equivalence theorem

Theorem *The statements (i) -(vi) are equivalent.*

Proof. Our proof uses two other equivalent claims (vii) and (viii) as given below.

Given a finite game $\Gamma_{CF} = \{N, v\}$, where each $v(S) = \{y \in \mathbf{R}^S \mid y \leq u_{jS}, \text{ all } j\}$ is defined by $k(S)$ corners u_{jS} , $j = 1, \dots, k(S)$. Let $K(S) = \{1, \dots, k(S)\}$, $Q > \text{Max}\{u_{ijs} \mid i \in S, j \in K(S), S \neq N\}$ be a large number; $m = \sum_{S \neq N} k(S)$, $C = C_{n \times m} = \{c_{kj}\}$ be given by: $c_k = c_{jS} = (u_{jS}, Qe_{-S})$ for each S and $j \in K(S)$; $A = A_{n \times m} = \{a_{kj}\} = \{e_{jS}\}$ be the incidence matrix for all S in C (i.e., $e_{jS} \equiv$

¹ This becomes the α -core existence in [8]. The general version of (vi) (Theorem 3 in [10]) implies (v) directly, while (vi) only implies (v) indirectly by the equivalence theorem in Section 2.

($e_S, 0_{-S}$), all $j \in K(S)$).² The following (vii), is equivalent to Theorem 2 in [6].

(vii) (Scarf, [6]) For the above A and C , $\exists x = (x_B, 0_{-B})$, $x_B \gg 0$ such that $Ax = e$, and that if we define $u_i = \text{Min}\{c_{ij} | j \in B\}$, then for each $k \in M = \{1, \dots, m\}$, $\exists i$ with $u_i \geq c_{ik}$.

The following (viii) reveals the geometry behind (v) and (vii). For each coalition $S \neq N$, let $\tilde{v}(S) = v(S) \times \mathbf{R}^{-S} \subset \mathbf{R}^n$. Let

$$(1) \quad GP(B) = \bigcap_{S \in B} \tilde{v}(S) \subset \mathbf{R}^n$$

be the n -dimensional payoffs generated by a balanced set B , and

$$(2) \quad GP = GP(\Gamma_{CF}) = \bigcup_{B \text{ balanced}} \bigcap_{S \in B} GP(B)$$

be the set of all generated payoffs. One sees that Γ_{CF} is balanced if $GP \subset v(N)$.

Since u is unblocked by $S \Leftrightarrow u \in [v(S) \setminus \partial v(S)]^C \times \mathbf{R}^S \subset \mathbf{R}^n$, where the superscript C denotes the complement set, the set of payoffs unblocked by all $S \neq N$ is given by

$$(3) \quad UBP = UBP(\Gamma_{CF}) = \bigcap_{S \neq N} \{[v(S) \setminus \partial v(S)]^C \times \mathbf{R}^S\},$$

and the core can alternatively be given by $C(\Gamma_{CF}) = \partial v(N) \cap UBP$.

(vii) is now equivalent to: a) the base B for x_B forms a balanced set;³ b) $u \in GP$; and c) $u \in UBP$ (by $u_i \geq c_{ik}$, all k). Hence, (vii) $\Leftrightarrow GP \cap UBP \neq \emptyset$ for finite games. Since any Γ_{CF} can be approximated by a sequence of finite games (§6 in [6]), one obtains (viii) below:

$$(viii) \quad GP(\Gamma_{CF}) \cap UBP(\Gamma_{CF}) \neq \emptyset \text{ holds for all } \Gamma_{CF}.$$

Now, come back to the proof for our theorem. Note that (ii) \Leftrightarrow (iii) \Leftrightarrow (vi) seems to be unknown to the public, although it existed already (see [10] for (ii) \Rightarrow (vi) \Rightarrow (iii); see [7] and p. 90 in [6] for (iii) \Rightarrow (vii) \Rightarrow (i)). By (i) \Leftrightarrow (ii) [3], (ii) \Rightarrow (iv) [1], (iv) \Rightarrow (ii) [9], (vii) \Rightarrow (v) [6], and by the above (vii) \Leftrightarrow (viii), our proof completes by showing (v) \Rightarrow (viii).

Since $v(S) \setminus \partial v(S)$ is open in \mathbf{R}^S , each $\{[v(S) \setminus \partial v(S)]^C \times \mathbf{R}^S\}$ is closed in \mathbf{R}^n . Hence,

$$(4) \quad UBP = \bigcap_{S \neq N} \{[v(S) \setminus \partial v(S)]^C \times \mathbf{R}^S\} = \bigcap_{B \text{ with } N \notin B} \left\{ \bigcap_{S \in B} \{[v(S) \setminus \partial v(S)]^C \times \mathbf{R}^S\} \right\}$$

² $c_k = c_{jS}$ and a_k , for unique $j \in K(S)$, are given by: $c_{ik} = u_{ijS}$, $a_{ik} \equiv 1$ if $i \in S$; $c_{ik} \equiv Q$, $a_{ik} \equiv 0$ if $i \notin S$.

³ Let $B = \{S \mid x_{jS} > 0 \text{ for some } j \in K(S)\}$, $\delta_S = \sum_{k \in S} x_k$ for each $S \in B$. One sees that B is balanced.

is a closed subset in \mathbf{R}^n . Let ∂GP be the upper surface of GP , \underline{GP}^C be the enclosure of GP^C , one has $\underline{GP}^C = \partial GP \cup GP^C$. By (1) and (2), one has

$$(5) \quad GP^C = \left\{ \bigcup_{B \text{ with } N \notin B} \left\{ \bigcap_{S \in B} [v(S) \times \mathbf{R}^S] \right\} \right\}^C = \bigcap_{B \text{ with } N \notin B} \left\{ \bigcup_{S \in B} [v(S)^C \times \emptyset^S] \right\}.$$

Let $\underline{v(S)}^C$ be the enclosure of $v(S)^C$. By $[v(S) \setminus \partial v(S)]^C = \underline{v(S)}^C$, one has

$$\bigcap_{S \in B} \{ [v(S) \setminus \partial v(S)]^C \times \mathbf{R}^S \} \subset \bigcup_{S \in B} [\underline{v(S)}^C \times \emptyset^S].$$

By (4), (5), and the above expression, one has

$$(6) \quad UBP = UBP(\Gamma_{CF}) \subset \underline{GP}^C.$$

Given (v), assume by way of contradiction that $GP \cap UBP = \emptyset$. By (6), UBP is included in the interior of \underline{GP}^C . Since ∂GP is the lower surface of \underline{GP}^C , one must have

$$(7) \quad d^* = d(\partial GP, UBP) = \text{Min} \{ \|x-y\| \mid x \in \partial GP, y \in UBP \} > 0.$$

Define a new game $\Gamma' = \{N, v(\cdot)'\}$ from Γ_{CF} by: $v(S)' = v(S)$, all $S \neq N$, and

$$(8) \quad v(N)' = GP(\Gamma_{CF}) \cup \left\{ \bigcup_{x \in \partial GP(\Gamma_{CF})} \{y \mid y \geq x \text{ and } d(x,y) < d^*/2\} \right\},$$

one has: $GP(\Gamma') = GP(\Gamma_{CF}) \subseteq v(N)'$, so Γ' is balanced. However, by (7)-(8) and by $UBP(\Gamma') = UBP(\Gamma_{CF})$, $d(\partial v(N)', UBP(\Gamma')) \geq d^*/2 > 0$, so $C(\Gamma') = \partial v(N)' \cap UBP(\Gamma') = \emptyset$, which contradicts to (v). Therefore, (viii) must hold. **Q.E.D.**

(Figure 2 about here)

3. Conclusions and open problems

This note has shown that the mathematical principles of fixed point theorems and the economic laws on equilibrium states are mirrored in each other. The equivalence theorem leaves with us three classes of problems for future research. The first class of problems are to find the equivalent versions of (iii)-(vi) for each of the more advanced fixed point theorems, and this line of research will provide a rich source of applications in economics, game theory and other social sciences for the fixed point literature. The second class of problems are to prove directly those relations in the equivalence that had been

indirectly established, these are marked as the dotted arrows in Figure 2. For example, it will be interesting to find a direct proof for $(iii) \Rightarrow (ii)$.⁴ The third class of problems are to apply the equivalence theorem in the growing literature on implementation theory and mechanism design, as each of the four social outcomes can be theoretically implemented by any of the other three social outcomes.

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⁴ Although $(ii) \Rightarrow (iii)$ earned John Nash the 1994 Nobel Prize in economics, it failed to impress John von Neumann, who "jumped ahead to the yet unstated conclusion" of (iii) and concluded "that is trivial," after Nash managed to state only a few disjoint sentences during their discussions (see p94 in [4]). One wonders how von Neumann would have responded had $(iii) \Rightarrow (ii)$ been known in that time.

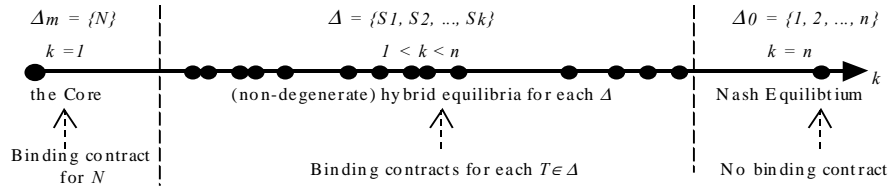


Figure 1. The spectrum of hybrid equilibria, where k ($1 \leq k \leq n$) is the number of coalitions in Δ .

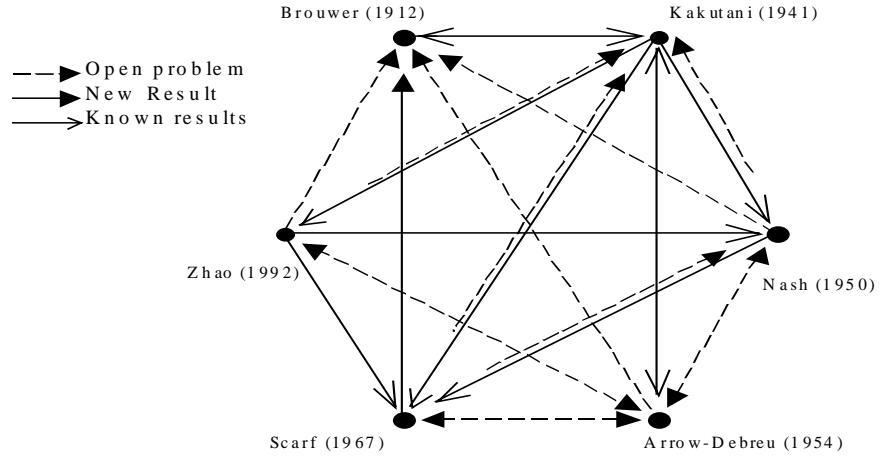


Figure 2. The equivalence between Brouwer' or Kakutani' fixed point theorem and the existence theorems of Nash equilibrium, competitive equilibrium, core and hybrid solution.