Econ 673: Microeconometrics
Chapter 10: Discrete/Continuous Choices

Discrete/Continuous Choice Models

- There are numerous settings in which the data are a combination of discrete and continuous choice variables jointly determined by underlying factors.
  - Housing type/housing expenditures
  - Appliance purchases/energy consumption
  - Electricity rate structure/energy consumption
  - Participation in educational programs/achievement results
  - Nonlinear budget constraints
- Jointly modeling the discrete and continuous choice variables is important to understand the impact of the exogenous factors

Outline

- Two examples of discrete/continuous decisions
  - Housing choice and the level of housing demand
    King, M. (1980), "An Econometric Model of Tenure Choice and Demand for Housing as a Joint Decision," *Journal of Public Economics* **14(2):** 137-159.
  - Appliance purchases and electricity demand
    Dubin, J. A., and D. L. McFadden (1984), "An Econometric Analysis of Residential Electric Appliance Holdings and Consumption," *Econometrica* 52(2): 345-362.
- Nonlinear budget constraints
- Chapter 12 will pick up a similar line of research using propensity score matching techniques

King (1980)

- Focuses on housing demand, jointly modeling
  - The choice between owner-occupied versus rental accommodations and
  - The level of housing expenditures
- Emphasizes that these choices are driven by the same set of underlying preferences
- Consistent estimates regarding the impact of changing exogenous factors must reflect the common driving forces
  - Tax changes
  - Access changes (rationing)

UK Housing Market

- Prior to WWI, housing in the UK was split
  - 90% privately rented
  - 10% owner-occupied
- As of 1977, this shifted substantially to
  - 13% privately rented
  - 56% owner occupied
  - 31% rented from local authorities
- King interested in understanding impacts of rationing (through mortgage and public housing access) on both the type of housing (i.e., housing "tenure") and level of housing expenditures

Model Structure

Individual households are assumed to solve the problem

$$\underset{x_1,x_2,j}{Max}\left[V\left(x_1,x_2,x_3^j;\beta\right) \mid p_1^j x_1 + p_2 x_2 \leq y,\ j \in \{1,2,3\}\right]$$

where

$V\left(x_1,x_2,x_3^j;\beta\right)$ denotes the direct utility function

x_1 denotes the quantity of housing purchased

x_2 denotes the quantity of other goods purchased

x_3^j denotes the level of security associate with housing type j

y denotes household income

The Decision is Segmented into 2 Stages

$$= \underset{j}{Max}\left[\left\{ \underset{x_1,x_2}{Max}\, V\left(x_1, x_2, x_3^j; \beta\right) \mid p_1^j x_1 + p_2 x_2 \le y \right\} \mid j \in \{1,2,3\} \right]$$

$$= \underset{j}{Max}\left[H\left(v_1, v_2; x_3^j, \beta\right) \mid j \in \{1,2,3\} \right]$$

where

$H\left(v_1, v_2; x_3^j, \beta\right)$ denotes conditional indirect utility

$v_k \equiv \dfrac{p_k}{y}$ denotes normalized prices

Conditional demands are then derived using Roy's identity

$$x_1^j = \frac{\partial H / \partial p_1^j}{\partial H / \partial y}$$

Functional Forms

King starts with a translog specification

$$-\ln H\left(v_1^j, v_2; x_3^j, \beta\right) = \beta_1 \ln v_1^j + \beta_2 \ln v_2 + \beta_3 \left(\ln v_1^j\right)^2$$
$$+ \beta_2 \left(\ln v_2\right)^2 + \beta_5 \left(\ln v_1^j\right)\left(\ln v_2\right) + \varepsilon^j$$

Captures unobserved security

Imposing homotheticity (for convenience) yields

$$u_n^j = \beta_1 \ln\left(\frac{p_{n1}^j}{p_{n2}}\right) + \beta_3 \left[\ln\left(\frac{p_{n1}^j}{p_{n2}}\right)\right]^2$$
$$+ \ln\left(\frac{y_n}{p_{n2}}\right) + \varepsilon_n^j$$

$$\varepsilon_n^j \sim N\left(0, \sigma_j^2\right), \sigma_j \equiv 1$$

Tenure Choice

Housing type choice then is determined by

$$\Pr[j] = \Pr\left[u_n^j > u_n^k \forall k \neq j\right]$$

Which is just a multivariate probit problem

Housing demand

Given the conditional indirect utility function

$$x_1^j = \frac{\partial H / \partial p_1^j}{\partial H / \partial y} + v_n^j$$

$$= \beta_1 \ln\left(\frac{y_n}{p_{n1}^j}\right) + 2\beta_3 \ln\left(\frac{y_n}{p_{n1}^j}\right) \ln\left(\frac{p_{n1}^j}{p_{n2}}\right) + v_n^j$$

$$v_n^j \sim N\left(0, \sigma_v^2\right)$$

Captures optimization errors

Assumes the error is from the same distribution regardless of j

Also assumes these errors are independent from ε_n^j

Estimation via ML

$$LL = \sum_{n=1}^{n}\sum_{j=1}^{n}\delta_{nj}\ln\Pr\left[j, x_1^j\right]$$

$$= \sum_{n=1}^{n}\sum_{j=1}^{n}\delta_{nj}\left\{\underbrace{\ln\Pr\left[j\right]}_{\text{Multivariate probit}} + \underbrace{\ln\Pr\left[x_1^j \mid j\right]}_{\text{Continuous normal LL}}\right\}$$

$$\delta_{nj} = \begin{cases} 1 & \text{if tenure type } j \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

Model Extensions

This basic framework is extended to allow for

- Cross-decision correlation by making the β's random
- Rationing
  - The probability of observing a household in tenure 1 is

$$p_{i1} = \hat{p}_{i1}q_{i1} + \hat{p}_{i2}\left(1 - q_{i2}\right)q_{i1}$$

where

$\hat{p}_{i1}$ denotes the unrationed probability of choosing tenure type 1

q_{ik} denotes the probability not being rationed when choosing type k

Data

- 5895 household data in 1973/74
- Housing quantities and prices are constructed using
  - rental rates (imputed for owner-occupied housing), controlling for differences in
    - tax treatment
    - housing appreciation
    - Subsidies in the public sector
    - Mortgage rates
  - Total housing expenditures

ML Estimates

	Fixed Coefficients	Fixed Coefficients Separate Estimation		Random Coefficients
β_1	0.0857	0.0417	0.0857	0.1022
β_3	0.0132	-1.029	0.0132	0.0238
q_1	0.704	0.844		0.705
q_2	0.818	0.847		0.818
LL	-7267	-6530		-6541

Price Elasticity of Housing Demand

Tenure Type	Fixed Coefficients	Random Coefficients
Owner-Occupied	-0.687	-0.523 (-0.240,-0.647)
Subsidized	-0.677	-0.498 (-0.176,-0.634)
Furnished rental	-0.744	-0.645 (-0.508,-0.718)

Dubin and McFadden (1984)

- Focuses on electricity consumption, jointly modeling
  - Appliance choice and
  - The level of electricity usage
- Interest driven by interest in modeling unit electricity consumption (UEC)

$$K_{iT} = \alpha_0 + \sum_{j=1}^{J} \delta_{ij} K_j + \varepsilon_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if household owns appliance } j \\ 0 & \text{otherwise} \end{cases}$$

concern centers around endogeneity of δ_{ij}

Portfolio Choice Model

Given a portfolio *i* of appliances

$$u = V\left(i, y - r_i, p_1, p_2, s_i, \varepsilon_i, \eta\right)$$

denotes the conditional indirect utility function, where

p_1 denotes the price of electricity

p_2 denotes the price of alternative energy resources

y denotes income

r_i denotes the price of portfolio i

s_i denotes observed attributes of portfolio i

ε_i denotes unobserved attributes of the portfolio i

η denotes unobserved attributes of the individual

Choices

$$\Pr\left\{\left(\varepsilon_1, \ldots, \varepsilon_m, \eta\right) \mid V\left(i, y - r_i, p_1, p_2, s_i, \varepsilon_i, \eta\right) > V\left(j, y - r_j, p_1, p_2, s_j, \varepsilon_j, \eta\right) \forall j \neq i\right\}$$

conditional on portfolio choice, consumption levels are determined by Roy's identity

$$x_1 = \frac{-\partial V\left(j, y - r_j, p_1, p_2, s_j, \varepsilon_j, \eta\right) / \partial p_1}{\partial V\left(j, y - r_j, p_1, p_2, s_j, \varepsilon_j, \eta\right) / \partial y}$$

$$x_2 = \frac{-\partial V\left(j, y - r_j, p_1, p_2, s_j, \varepsilon_j, \eta\right) / \partial p_2}{\partial V\left(j, y - r_j, p_1, p_2, s_j, \varepsilon_j, \eta\right) / \partial y}$$

Alternative Approach

Dubin and McFadden suggest the alternative approach of

- specifying UEC equations
- treating Roy's identity as a partial differential equation and solving back for conditional indirect utility function

$$x_1 = \beta_i \left(y - r_i \right) + m^i \left(p_1, p_2 \right) + v_{1i}$$

$$\Rightarrow u = \psi \left(\left[M^i \left(p_1, p_2 \right) + y - r_i + v_{1i} / \beta_i \right] e^{-\beta_i p_1}, p_2, v_{2i} \right)$$

where

$$M^i \left(p_1, p_2 \right) = \int_{p_1}^{0} m^i \left(p_1, p_2 \right) e^{\beta_i (p_1 - t)} dt$$

Restrictions

Several alternative restrictions are considered

Version #1 $\quad v_{2i} = v_{21} \forall i$

$$\Rightarrow u = \psi \left(\left[M^i \left(p_1, p_2 \right) + y - r_i + v_{1i} / \beta_i \right] e^{-\beta_i p_1}, p_2, v_{21} \right)$$

$$\Pr \left\{ \left(\varepsilon_1, \ldots, \varepsilon_m, \eta \right) | V \left(i, y - r_i, p_1, p_2, s_i, \varepsilon_i, \eta \right) > V \left(j, y - r_j, p_1, p_2, s_j, \varepsilon_j, \eta \right) \forall j \neq i \right\}$$

$$P_i = \Pr \left\{ \left[M^i \left(p_1, p_2 \right) + y - r_i + v_{1i} / \beta_i \right] e^{-\beta_i p_1} > \left[M^j \left(p_1, p_2 \right) + y - r_j + v_{1j} / \beta_j \right] e^{-\beta_j p_1} \forall j \neq i \right\}$$

In this case, portfolio choice is driven entirely by v_{1i}

Restrictions (cont'd)

If further

$$u = \ln\left\{\left[\alpha_0^i + \frac{\alpha_1^i}{\beta} + \alpha_1^i p_1 + \alpha_2^i p_2 + \beta\left(y - r_i\right) + v_{1i}\right] e^{-\beta p_1}\right\} - \alpha_5 \ln p_2$$

then

$$x_1 = \alpha_0^i + \alpha_1^i p_1 + \alpha_2^i p_2 + \beta\left(y - r_i\right) + v_{1i}$$

Implies income impact on electricity consumption is independent of the appliance portfolio

Restrictions (cont'd)

Version #2 $v_{1i} = \eta$

and

$$u = \left[\alpha_0^i + \frac{\alpha_1^i}{\beta} + \alpha_1^i p_1 + \alpha_2^i p_2 + \beta\left(y - r_i\right) + \eta\right] e^{-\beta p_1} - \alpha_5 \ln p_2 + v_{2i}$$

then

$$x_1 = \alpha_0^i + \alpha_1^i p_1 + \alpha_2^i p_2 + \beta\left(y - r_i\right) + \eta$$

and

$$P_i = \Pr\left\{v_{2j} - v_{2i} < W_i - W_j \forall j \neq i\right\}$$

$$W_i \equiv \left[\alpha_0^i + \frac{\alpha_1^i}{\beta} + \alpha_1^i p_1 + \alpha_2^i p_2 + \beta\left(y - r_i\right) + \eta\right] e^{-\beta p_1}$$

Application

In empirical analysis, a variant of version #2 is used

$$u = \left[\alpha_0^i + \frac{\alpha_1}{\beta} + \alpha_1 p_1 + \alpha_2 p_2 + w'\gamma + \beta \left(y - r_i \right) + \eta \right] e^{-\beta p_1} + \varepsilon_i$$

will not impact choice probabilities

where w is a vector of household characteristics

ε_i ~iid extreme value

Only two portfolios are considered:

1: Both electric space and electric water heating

2: Neither electric space nor electric water heating

Application

yields

$$x_1 = \alpha_0^0 + \delta_1 \left(\alpha_0^1 - \alpha_0^0 \right) + \alpha_1 p_1 + \alpha_1 p_2 + w'\gamma + \beta \left(y - \delta_1 r_1 \right) + \eta$$

$$\delta_1 = \begin{cases} 1 & \text{electric portfolio is chosen} \\ 0 & \text{otherwise} \end{cases}$$

Two-step estimation:

Step 1: Portfolio choice

Step 2: Demand estimation using instruments for δ_1

Price and Income Elastiticities

		Least Squares	IV
All Electric	Income	0.028	0.008
	Electric Price	-0.197	-0.310
	Gas Price	-0.033	-0.013
No Electric	Income	0.079	0.022
	Electric Price	0.021	0.042
	Gas Price	-0.093	-0.037
Including Portfolio Shift	Income	0.06	0.02
	Electric Price	-0.22	-0.26
	Gas Price	0.35	0.039

Nonlinear Budget Constraints

- While theory relies largely on the simplifying assumption that the budget constraint is linear, it is often nonlinear in practice
- Examples:
  - Labor supply, due to
    - tax laws
    - social security and disability benefits
    - food stamps, etc.
  - Nonlinear pricing
    - Inverted block rates
    - Declining block rates
  - Season Tickets
  - Non-linear interest rates

Sources

- Hausman, J. (1985), "The Econometrics of Nonlinear Budget Sets," *Econometrica* **53**(6): 1255-1282.
- *Herriges, J., and K.K. King (1994), "Residential Demand for Electricity Under Inverted Block Rate: Evidence for a Controlled Experiment," *Journal of Business and Economic Statistics* **12**(4): 419-430.
- *Pudney, S. (1989), *Modeling Individual Choice: The Econometrics of Corners, Kinks, and Holes*, Cambridge, MA: Basil Blackwell, Ch. 5

Conventional Approach

In most theoretical specifications of consumer choice, an individual is assumed to solve

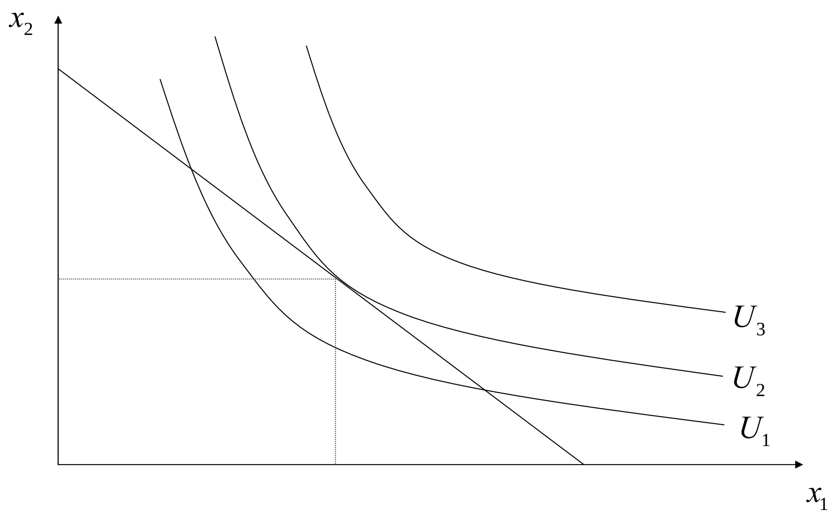
$$\underset{x}{Max}\; U\left(x\right) \text{ s.t. } p'x \le y$$

where

x is a $k \times 1$ vector of commodities
p is a $k \times 1$ vector of corresponding prices

The implicit assumption here is that the unit cost of each commodity is independent of the quantity consumed

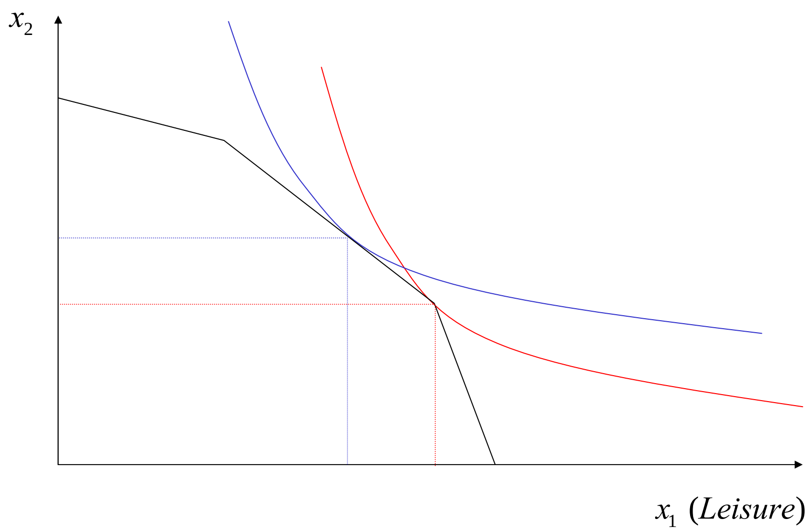
Graphically



Convex Budget Sets

- Convex budget sets arise in many settings in which income redistribution is a goal; e.g.,
  - progressive income taxes
  - low cost subsistence provisions of
    - water
    - electricity
    - heat
  - Energy tariffs (designed to discourage consumption)

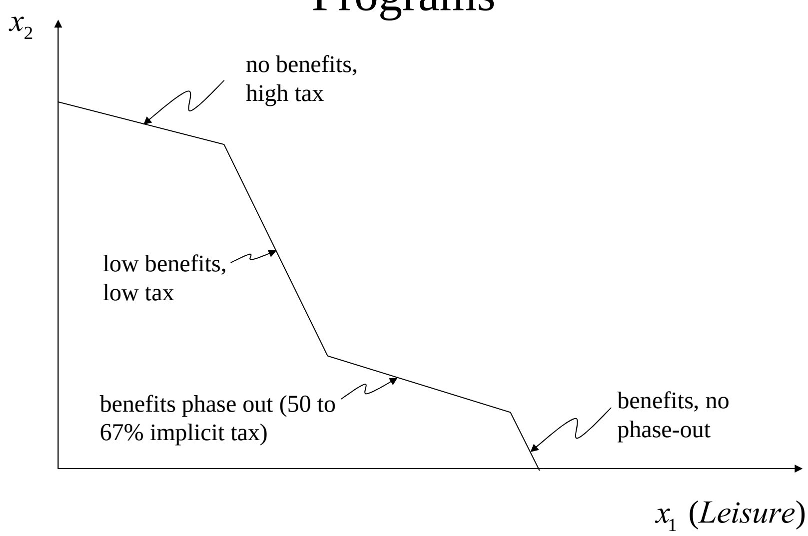
Example #1: Strictly Progressive Taxes



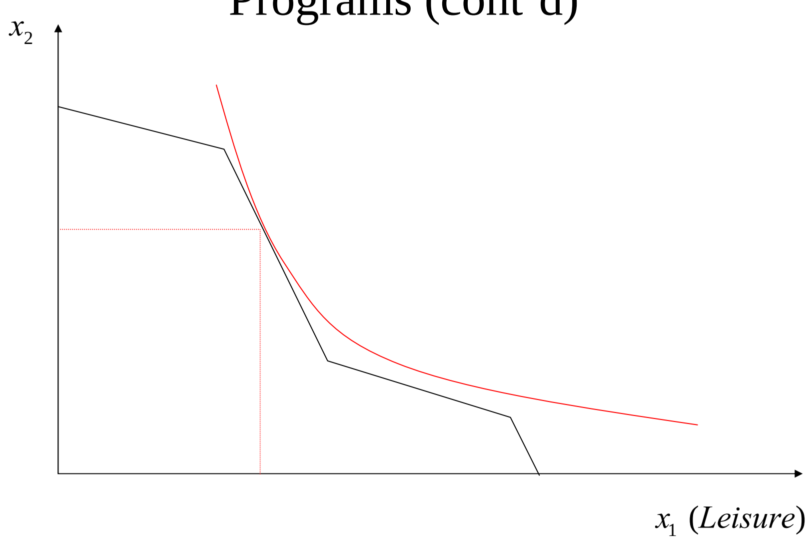
Nonconvex Budget Sets

- Nonconvexities are also common due to
  - Bulk discounts
  - Season passes
  - Tax loopholes
  - Phased out welfare programs

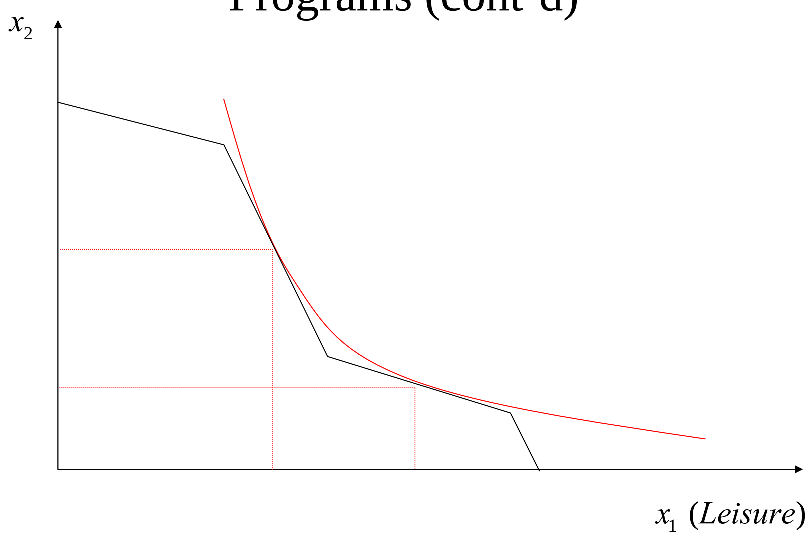
Example #2: Phased-out Welfare Programs



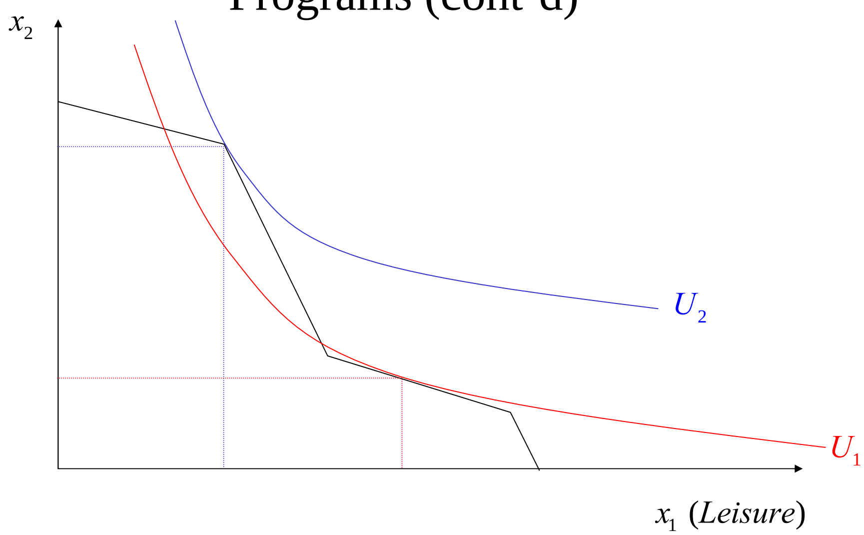
Example #2: Phased-out Welfare Programs (cont'd)



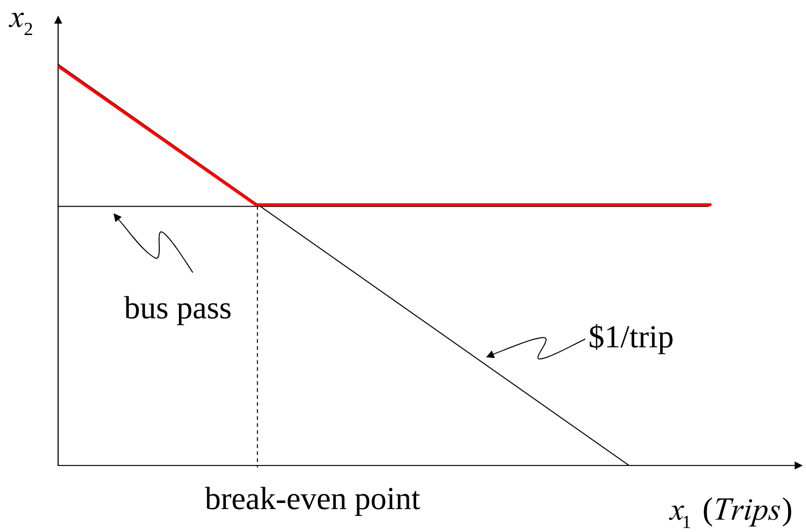
Example #2: Phased-out Welfare Programs (cont'd)



Example #2: Phased-out Welfare Programs (cont'd)



Example #3: Monthly Bus Pass



Complex Budget Constraints
UK Tax-Social Security System
Pudney (1989)

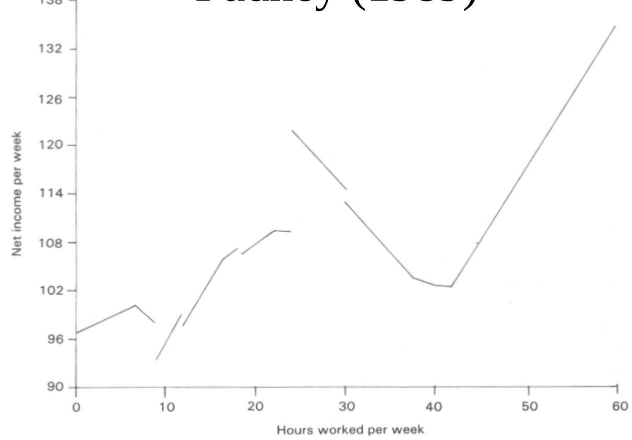


Figure 5.5 The work–income frontier for a hypothetical low-income family.

The General Solution

Hausman (1985) suggests viewing the solution to the consumer's problem as a sequential one

- Finding the optimal consumption bundle along a given budget segment
- Choosing that segment maximizing overall utility

$$V_j\left(p_j, y_j, q_j\right) = \underset{x}{Max}\left\{U(x) : c_j\left(x, p_j, q_j, y_j\right) \leq 0, x \geq 0\right\}$$

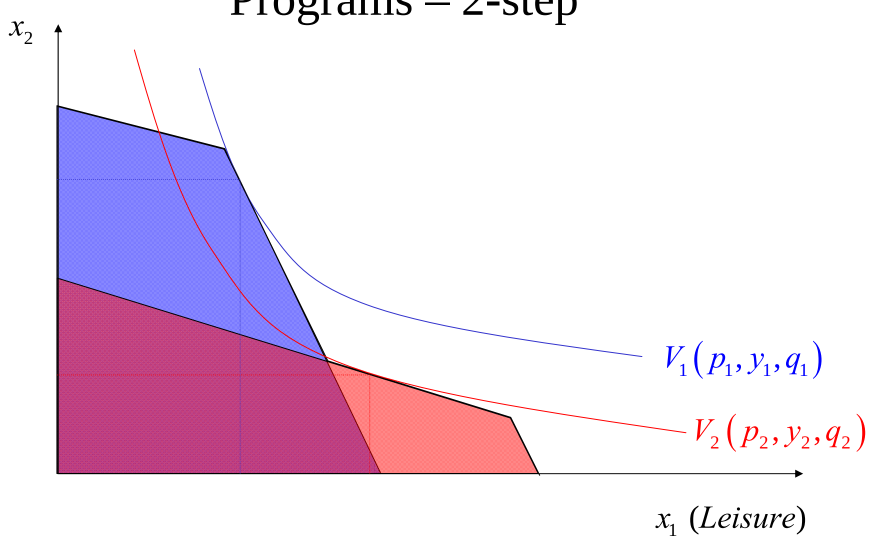
$$V^*\left(\{p\}, y, \{q\}\right) = \underset{j}{Max}\left\{V\left(p_j, q_j, y_j\right), j = 1, \ldots, J\right\}$$

where

$c_j\left(x, p_j, q_j, y_j\right) \leq 0$ defines a convex portion of the budget constraint

the q_i's are quantities used in defining these portions

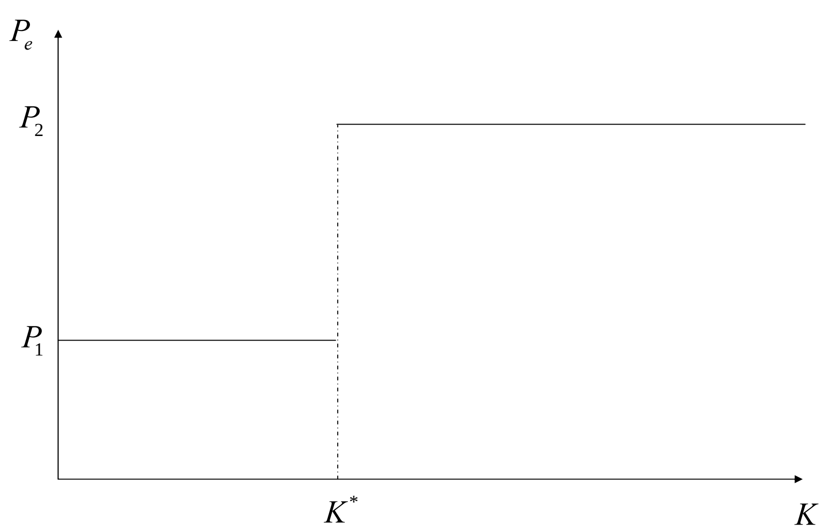
Example #2: Phased-out Welfare Programs – 2-step



Application: Inverted Block Rates Herriges and King (1994)

- Authors estimate residential demand for electricity under inverted block rates
  - Price follows a step-function
    - initial usage has a low marginal rate to insure subsistence level of energy for low income households
    - usage beyond subsistence level faces a higher marginal price
  - Analysis is based on data from a pricing experiment
- The paper provides
  - estimates based on a variety of econometric procedures
  - both parameter estimates and demand elasticities

Inverted Block Rates



The Fundamental Problem

- Under inverted block rates, the marginal price of electricity is given by

$$P_e\left(K; P_1, P_2, K^*\right) = \begin{cases} P_1 & K \le K^* \\ P_2 & K > K^* \end{cases}$$

- The budget constraint then takes the form

$$P_2 K - \left(P_2 - P_1\right)\min\left(K, K^*\right) + P_g G \le Y$$

or equivalently

$$P_2 K + P_g G \le Y + \left(P_2 - P_1\right)\min\left(K, K^*\right)$$

The Fundamental Problem (cont'd)

- Consider a simple linear demand specification

$$\begin{aligned} K &= \alpha + \beta P_e + \gamma Y + \varepsilon \\ &= \alpha + \beta P_e\left(K; P_1, P_2, K^*\right) + \gamma Y + \varepsilon \end{aligned}$$

the problem is that price is endogenously determined

- OLS will be biased in a positive direction under inverted block rates – estimating in part the supply relationship rather than demand

Alternative Solutions

- Reduced Form
- Instrumental Variables
- Structural Maximum Likelihood (SML)
- Modified Structural Maximum Likelihood (MSML)

Reduced Form Estimation

- Reduced form estimation models demand as a function (typically linear) of the elements defining the block rate structure or some index of these components
- For example, one might estimate

$$K = \tilde{\alpha} + \left[\tilde{\beta}_1 P_1 + \tilde{\beta}_2 P_2 + \tilde{\beta}_* K^* \right] + \tilde{\gamma} Y + \varepsilon$$

- Alternatively, it has been suggested that one use

$$K = \tilde{\alpha} + \left[\overline{\beta}_1 \overline{P} + \beta_2 P_2 \right] + \tilde{\gamma} Y + \varepsilon$$

i.e., using average and marginal price terms

Merits of Reduced Form Approach

- Primary advantage: Eliminates endogeneity of price terms
- Disadvantages:
  - Limits the ability of the model to forecast the impact of changing rate structure components
    - Suppose K^* is fixed in available data
    - The resulting model cannot be used to forecast the impact of changing K^*
  - Models are typically ad hoc, with little theoretical justification
  - Parameters can be difficult to interpret

Instrumental Variables (IV) Approach

- A linear demand specification could be written as

$$K = \alpha + \beta P_e\left(K; P_1, P_2, K^*\right) + \gamma Y_v\left(K; P_1, P_2, K^*\right) + \varepsilon$$

where

$$P_e\left(K; P_1, P_2, K^*\right) = \begin{cases} P_1 & K \le K^* \\ P_2 & K > K^* \end{cases}$$

and

$$Y_v\left(K; P_1, P_2, K^*\right) = \begin{cases} Y & K \le K^* \\ Y + \left(P_2 - P_1\right)K^* & K > K^* \end{cases}$$

Instrumental Variables (IV) Approach (cont'd)

- The reduced form model could be used to form instruments for both the marginal price and the virtual income, using

$$\hat{P}_e = \Pr\left(P_e = P_1 \mid Y\right) P_1 + \Pr\left(P_e = P_2 \mid Y\right) P_2$$

where

$$\hat{Y}_v = Y + \Pr\left(P_e = P_2 \mid Y\right)\left(P_2 - P_1\right) K^*$$

and then estimating

$$K = \alpha + \beta \hat{P}_e + \gamma \hat{Y}_v\left(K; P_1, P_2, K^*\right) + \tilde{\varepsilon}$$

Relative Merits of IV

- Superior to RF in several respects
  - explicitly accounts for endogeneity of the price signal
  - retains neoclassical framework in deriving demand equations, allowing clearer linkages back to indirect utility function for welfare analysis
- Limitations:
  - Inefficient, ignoring obvious linkages between price determination and demand determination
  - One may be limited in the ability to forecast demand changes due to price component changes

Structural Maximum Likelihood (SML)

- The SML approach was introduced by Burtless and Hausman (1978) in modeling the impact of taxation on labor supply
- It has subsequently been used to model
  - Aid to Families with Dependent Children (Moffitt, 1984)
  - Charitable Contributions (Reece and Zieschang, 1985)
  - Electricity Demand (Dubin, 1985)
  - Disability Insurance (Hausman, 1985)
  - Labor Supply (Arrufat and Zabalza, 1986)
- It provides a unified model of both
  - the demand given a specific price segment
  - the selection of the price segment along which to consume

The Basic SML Model

Individual consumer are assumed to solve

$$\underset{K,G}{Max}\ U\left(K,G,\varepsilon\right)$$

s.t.

$$P_1 K + G \le Y$$

$$P_2 K + G \le Y + \left(P_2 - P_1\right) K^*$$

where

ε captures random heterogeneity of preferences among consumers

The Basic SML Model (cont'd)

The corresponding Lagrangian is given by

$$L = U\left(K, G, \varepsilon\right) - \lambda_1 \left[Y - P_1 K - G\right] - \lambda_2 \left[Y + \left(P_2 - P_1\right) K^* - P_2 K - G\right]$$

with first order conditions

$$0 = \frac{\partial L}{\partial K} = U_K\left(K, G, \varepsilon\right) + \lambda_1 P_1 + \lambda_2 P_2$$

$$0 = \frac{\partial L}{\partial G} = U_G\left(K, G, \varepsilon\right) + \lambda_1 + \lambda_2$$

$$0 = \lambda_1 \left[Y - P_1 K - G\right]$$

$$0 = \lambda_2 \left[Y + \left(P_2 - P_1\right) K^* - P_2 K - G\right]$$

Three possibilities emerge

Region I ($K<K^*$)

Suppose only the first budget constraint is binding; i.e.,

$$P_1 K + G \le Y$$

then

$$K < K^*, \lambda_2 = 0$$

and the first order conditions become

$$0 = \frac{\partial L}{\partial K} = U_K\left(K, G, \varepsilon\right) + \lambda_1 P_1$$

$$0 = \frac{\partial L}{\partial G} = U_G\left(K, G, \varepsilon\right) + \lambda_1$$

$$0 = \lambda_1 \left[Y - P_1 K - G\right]$$

Region I ($K<K^*$) (cont'd)

But these are simply the first order conditions associated with the problem

$$\underset{K,G}{Max}\ U\left(K,G,\varepsilon\right) \quad \text{s.t.} \quad P_1K+G\le Y$$

with corresponding demand equations

$$K=F\left(P_1,Y,\varepsilon\right)$$

Thus, for $K<K^*$, our solution set is

$$S_I=\left\{F\left(P_1,Y,\varepsilon\right),Y-P_1F\left(P_1,Y,\varepsilon\right)\right\}$$

Region II ($K>K^*$)

Suppose only the second budget constraint is binding; i.e.,

$$P_2K+G\le Y+\left(P_2-P_1\right)K^*$$

i.e., we are on the second segment of the budget constraint

then

$$K>K^*,\lambda_1=0$$

and the first order conditions become

$$0=\frac{\partial L}{\partial K}=U_K\left(K,G,\varepsilon\right)+\lambda_2P_2$$
$$0=\frac{\partial L}{\partial G}=U_G\left(K,G,\varepsilon\right)+\lambda_2$$
$$0=\lambda_2\left[Y+\left(P_2-P_1\right)K^*-P_2K-G\right]$$

Region II (K>K^*) (cont'd)

But these are simply the first order conditions associated with the problem

$$\underset{K,G}{Max}\ U\left(K,G,\varepsilon\right) \quad \text{s.t.} \quad P_2K + G \leq Y + \left(P_2 - P_1\right)K^*$$

with corresponding demand equations

$$K = F\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right]$$

Thus, for K>K^*, our solution set is

$$S_{II} = \left\{F\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right], Y - P_2\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right]\right\}$$

Region III (K=K^*)

Finally, suppose both budget constraints are binding, then

$$K = K^*$$

i.e., we are at the kink in the budget constraint

and our solution set becomes

$$S_{III} = \left\{K^*, Y - P_1K^*\right\}$$

Notice that this solution set is independent of the error term ε

Three-Part Demand Equation

The resulting demand equation is given by the three-part function

$$K = \begin{cases} F\left[P_1, Y, \varepsilon\right] & F\left[P_1, Y, \varepsilon\right] < K^* \\ K^* & F\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right] \le K^* \le F\left[P_1, Y, \varepsilon\right] \\ F\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right] & F\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right] > K^* \end{cases}$$

An attractive feature of this approach is that we can use the same functional form for the demand equation for the different segments

Three-Part Demand Equation (cont'd)

Define $\varepsilon_i = \varepsilon_i(P)$ as implicitly solving

$$F\left[P_i, Y + \left(P_i - P_1\right)K^*, \varepsilon_i\right] = K^*$$

Assume that

$$\frac{\partial F\left[P, Y, \varepsilon\right]}{\partial \varepsilon} > 0$$

We can then write

$$K = \begin{cases} F\left[P_1, Y, \varepsilon\right] & \varepsilon < \varepsilon_1 \\ K^* & \varepsilon_1 \le \varepsilon \le \varepsilon_2 \\ F\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right] & \varepsilon > \varepsilon_2 \end{cases}$$

Three-Part Demand Equation (cont'd)

Note that

$$\Pr\left[K = K^*\right] = \Pr\left[\varepsilon_1 \le \varepsilon \le \varepsilon_2\right]$$

One implication of this specification is that we should observe a point mass at the kink, which we typically do not

One rational for the lack of a point mass at K^* is that there is also measurement error in the data due to

- meter malfunctions
- missed meter readings
- rebills, etc.

Introducing Measurement Error

Suppose measured usage is related to actual usage via

$$K_m = K_a \eta$$

$$= \begin{cases} F\left[P_1, Y, \varepsilon\right]\eta & \varepsilon < \varepsilon_1 \\ K^* \eta & \varepsilon_1 \le \varepsilon \le \varepsilon_2 \\ F\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right]\eta & \varepsilon > \varepsilon_2 \end{cases}$$

Thus

$$\Pr\left(K_m = K_0\right) = \Pr\left[K_0 = F\left(P_1, Y, \varepsilon\right)\eta, \varepsilon < \varepsilon_1\right]$$
$$+ \Pr\left[K_0 = K^* \eta, \varepsilon_1 \le \varepsilon \le \varepsilon_2\right]$$
$$+ \Pr\left[K_0 = F\left(P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right)\eta, \varepsilon > \varepsilon_2\right]$$

The Modified SML

A problem with the SML approach is that it assumes that

$$\varepsilon_1 < \varepsilon_2$$

This turns out to be true if preferences are convex over a specified interval, but need not hold in general

One could

- impose convex preferences globally, but this imposes considerable structure on preferences
- impose convexity within selected region of the data space – difficult to do with nonlinear budget constraints

The Modified SML (cont'd)

We modified the demand specification, taking into account the possibility the $\varepsilon_1 > \varepsilon_2$

$$K_m = \begin{cases} F\left[P_1, Y, \varepsilon\right]\eta & \varepsilon < \min\left(\varepsilon_1, \varepsilon_2\right) \\ K^*\eta & \varepsilon \in T\left(\varepsilon_1, \varepsilon_2\right) \\ F\left[P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right]\eta & \varepsilon > \max\left(\varepsilon_1, \varepsilon_2\right) \end{cases}$$

where

$$T\left(a,b\right) \equiv \left\{x \mid a \le x \le b\right\}$$

The Modified SML (cont'd)

The resulting probabilities become

$$\Pr\left(K_m = K_0\right) = \Pr\left[K_0 = F\left(P_1, Y, \varepsilon\right)\eta, \varepsilon < \min\left(\varepsilon_1, \varepsilon_2\right)\right]$$
$$+\Pr\left[K_0 = K^*\eta, \varepsilon \in T\left(\varepsilon_1, \varepsilon_2\right)\right]$$
$$+\Pr\left[K_0 = F\left(P_2, Y + \left(P_2 - P_1\right)K^*, \varepsilon\right)\eta, \varepsilon > \max\left(\varepsilon_1, \varepsilon_2\right)\right]$$

from which the corresponding log-likelihood function can be derived, given assumptions on ε and η.

We assumed that these error components were independent normal and log-normal respectively

Application

- Data from Wisconsin Electric's Residential Rate Experiment
- Participants were randomly selected from WE's service territory, excluding households
  - with two or more accounts
  - from master-metered dwellings
  - with usage in excess of 1500 kWh/month

  these exclusions accounted for <5% of the population
- Surveys were periodically conducted to elicit household characteristics

Application (cont'd)

- Households were randomly assigned to one of five tariffs, including
  - four inverted block rates (INV)
  - one flat rate tariff

Rate	P_1	P_1	K^*	Number of Customers
150	4.01	9.40	250	63
151	2.51	10.38	250	50
152	4.01	13.74	500	54
153	2.51	16.71	500	60
156	6.63	6.63	n.a.	143

Parameter Estimates

Parameter	Summer	Winter
α	7.45 (0.03)	7.51 (0.03)
β	-0.02 (0.08)	-0.04 (0.20)
δ	0.46 (0.05)	0.44 (0.04)
σ_ε	0.31 (0.31)	0.54 (0.20)
σ_η	0.41 (0.23)	0.01 (<0.01)

Elasticities

- The price elasticities represent short-run responses to price changes, but are somewhat smaller than found in previous studies
- The income elasticities are typical
- One advantage of the SML and MSML approaches is that we can construct elasticities with respect to components of the price schedule

Expected Demand

$$
\begin{aligned}
\hat{K} &= E\left[K_m\right] \\
&= \left\{ \exp\left(F_{k1}\right)\exp\left[\frac{\left(\sigma_\tau^2+\sigma_\varepsilon^2\right)}{2}\right]\Phi\left[\left\{\frac{\ln\left(K^*\right)-F_{k1}}{\sigma_\varepsilon}\right\}-\sigma_\varepsilon\right]\right. \\
&\quad +K^*\exp\left(\frac{\sigma_\varepsilon^2}{2}\right)\left[\Phi\left\{\frac{\ln\left(K^*\right)-F_{k2}}{\sigma_\varepsilon}\right\}-\Phi\left\{\frac{\ln\left(K^*\right)-F_{k1}}{\sigma_\varepsilon}\right\}\right] \\
&\quad \left.+\exp\left(F_{k2}\right)\exp\left[\frac{\left(\sigma_\tau^2+\sigma_\varepsilon^2\right)}{2}\right]\left(1-\Phi\left[\left\{\frac{\ln\left(K^*\right)-F_{k1}}{\sigma_\varepsilon}\right\}-\sigma_\varepsilon\right]\right)\right\}D \\
&\quad +\left\{\exp\left(F_{k1}\right)\exp\left[\frac{\left(\sigma_\tau^2+\sigma_\varepsilon^2\right)}{2}\right]\Phi\left[\left\{\frac{\ln\left(K^*\right)-F_{k1}}{\sigma_\varepsilon}\right\}-\sigma_\varepsilon\right]\right. \\
&\quad \left.+\exp\left(F_{k2}\right)\exp\left[\frac{\left(\sigma_\tau^2+\sigma_\varepsilon^2\right)}{2}\right]\left(1-\Phi\left[\left\{\frac{\ln\left(K^*\right)-F_{k1}}{\sigma_\varepsilon}\right\}-\sigma_\varepsilon\right]\right)\right\}\left(1-D\right) \\
&\quad /\left\{1+\Phi\left\{\frac{\ln\left(K^*\right)-F_{k2}}{\sigma_\varepsilon}\right\}-\Phi\left\{\frac{\ln\left(K^*\right)-F_{k1}}{\sigma_\varepsilon}\right\}\right\}
\end{aligned}
$$

Expected Demand Elasticities

We were interested in constructing the following elasticities for expected demand

$$\eta_{P_1} = \frac{\partial \ln \hat{K}}{\partial \ln P_1}$$

$$\eta_{P_1} = \frac{\partial \ln \hat{K}}{\partial \ln P_2}$$

$$\eta_{P_1} = \frac{\partial \ln \hat{K}}{\partial \ln K^*}$$

Elasticities with Respect to P_1

Annual Income	150	151	152	153
$3000	-0.0244	-0.0199	-0.0386	-0.0392
$27,500	-0.0025	-0.0018	-0.0103	-0.0090
$90,000	-0.0006	-0.0004	-0.0024	-0.0019

Elasticities with Respect to P_2

Annual Income	150	151	152	153
$3000	0.0011	0.0031	-0.0032	-0.0033
$27,500	-0.0326	-0.0321	-.00194	-0.0172
$90,000	-0.0363	0.0362	-0.0324	-0.0315

Elasticities with Respect to K^*

Annual Income	150	151	152	153
$3000	0.0204	0.0342	0.0212	0.0316
$27,500	0.0048	0.0074	0.0265	0.0403
$90,000	0.0010	0.0015	0.0091	0.0137

Comparison of Alternative Methods

	OLS	RF	IV	MSML
α	7.38	7.45	7.41	7.45
β	0.44		-0.05	-0.02
$\tilde{\beta}_1$		0.06		
$\tilde{\beta}_2$		0.08		
$\tilde{\beta}_*$		0.08		
δ	0.36	0.45	0.24	0.46