**ECON 653 - HOMEWORK 4**
(Due: October 13, 2009)

1. Consider a mean-variance investor with utility $U[E(\tilde{Y}), Var(\tilde{Y})] = E(\tilde{Y}) - \alpha/2\ Var(\tilde{Y})$, where $\alpha > 0$, $E(\cdot)$ and $Var(\cdot)$ denote the expectation and the variance operators, and $\tilde{Y}$ is terminal wealth. The individual is endowed with $\underline{Q}_1$ units of asset 1 and $\underline{Q}_2$ units of asset 2. Each unit of assets 1 and 2 has a payoff of $\tilde{X}_1$ and $\tilde{X}_2$, respectively, and the correlation between $\tilde{X}_1$ and $\tilde{X}_2$ is $\rho$. Asset 1 has a price of $P_1$, and asset 2 is the numeraire (i.e., fix $P_2 = 1$). Hence, the investor's initial wealth is $Y_0 = P_1\ \underline{Q}_1 + \underline{Q}_2$.
   a. Solve for $Q_i^*(P_1; \alpha, E(\tilde{X}_1), E(\tilde{X}_2), Var(\tilde{X}_1), Var(\tilde{X}_2), \rho, \underline{Q}_1, \underline{Q}_2)$, $i = 1, 2$, the optimal number of units to invest in asset $i$ as a function of the underlying prices and parameters (i.e., function $Q_i^*(\cdot)$ is the investor's demand for asset $i$).
   b. Use Excel to make graphs of $Q_1^*(\cdot)$ as a function of $P_1$, $\alpha$, $E(\tilde{X}_1)$, $E(\tilde{X}_2)$, $Var(\tilde{X}_1)$, $Var(\tilde{X}_2)$, $\rho$, $\underline{Q}_1$, and $\underline{Q}_2$. (Note: To follow standard convention, for the $Q_1$-$P_1$ plot you should put $Q_1$ on the horizontal axis and $P_1$ on the vertical axis.) For the variables that are held constant in each graph, use the following values: $P_1 = 2$, $\alpha = 1$, $E(\tilde{X}_1) = 4$, $E(\tilde{X}_2) = 1.2$, $Var(\tilde{X}_1) = 0.16$, $Var(\tilde{X}_2) = 0.01$, $\rho = 0$, $\underline{Q}_1 = 3$, and $\underline{Q}_2 = 5$.
   c. Suppose that the economy just described is inhabited by $I$ investors with identical utility functions and endowments. Is $P_1 = 2$ an equilibrium price? Why? Find the equilibrium price function $P_1^{eq}(\cdot)$. (Note: You only need to provide an implicit solution for $P_1^{eq}(\cdot)$, as you may not be able to obtain an explicit solution.) Using the parameterization given in Problem 1.b, obtain a numerical solution for $P_1^{eq}$ (Note: You may use **Tools Solver** in Excel to perform the requested calculation.).
   d. Suppose now that the economy is inhabited by $I$ investors with identical endowments, but $I/2$ investors have $\alpha = 1$ and $I/2$ investors have $\alpha = 4$. Is $P_1 = 2$ an equilibrium price? Why? Find the equilibrium price function $P_1^{eq}(\cdot)$, and obtain a numerical solution for $P_1^{eq}$. How does this numerical value compare with the solution you obtained in Problem 1.c? Is your finding reasonable?

2. Express the optimization problem described in Problem 1 above in terms of portfolio shares and returns ($\tilde{R}_i \equiv \tilde{X}_i / P_i$).
   a. Solve for $w_i^*(\alpha, Y_0, E(\tilde{R}_1), E(\tilde{R}_2), Var(\tilde{R}_1), Var(\tilde{R}_2), \rho)$, $i = 1, 2$, the optimal share of wealth invested in asset $i$ as a function of the underlying parameters.
   b. Use Excel to make graphs of $w_1^*(\cdot)$ as a function of $\alpha$, $Y_0$, $E(\tilde{R}_1)$, $E(\tilde{R}_2)$, $Var(\tilde{R}_1)$, $Var(\tilde{R}_2)$, and $\rho$.
   c. Suppose that the economy just described is inhabited by $I$ investors with identical utility functions and endowments. Is $\tilde{R}_1 = \tilde{X}_1/2$ an equilibrium return? Why? Find the equilibrium return function $\tilde{R}_1^{eq}(\cdot)$ and obtain a numerical solution for $\tilde{R}_1^{eq}$. (Note: Please take into account that $Y_0$ is a function of $P_1$.)
   d. Plot the efficient frontier and the investor's indifference curves in mean-standard deviation space for both $\tilde{R}_1 = \tilde{X}_1/2$ and $\tilde{R}_1^{eq}$. Calculate the return on the market portfolio in equilibrium.
   e. Suppose now that the economy is inhabited by $I$ investors with identical utility functions, but $I/2$ investors have endowments $(\underline{Q}_1, \underline{Q}_2) = (3, 5)$ and $I/2$ investors have endowments $(\underline{Q}_1, \underline{Q}_2) = (4, 1)$. Is $\tilde{R}_1 = \tilde{X}_1/P_1$ an equilibrium return? Why? Find the equilibrium return function $\tilde{R}_1^{eq}(\cdot)$, and obtain a numerical solution for $\tilde{R}_1^{eq}$. How does this numerical value compare with the solution you obtained in Problem 2.c? Is your finding reasonable? Calculate the return on the market portfolio in equilibrium.

3. Consider the setup of Problem 1, but assuming that there is also a risk-free asset with payoff $X_f$ = 0.65 and price $P_f$ = 0.5, and that the agent is endowed with $\underline{Q}_f$ = 4 units of it.
   a. Suppose that the economy just described is inhabited by *I* investors with identical utility functions and endowments. Are $P_1$ = 2 and $P_f$ = 0.5 equilibrium prices? Why? Obtain numerical solutions for $P_1^{eq}$ and $P_f^{eq}$.
   b. Express the optimization problem in terms of portfolio shares and returns. Suppose that the economy is inhabited by *I* investors with identical utility functions and endowments. Are $\tilde{R}_1$ = 0.5 $\tilde{X}_1$ and $R_f$ = 1.3 equilibrium returns? Why? Obtain numerical solutions for $\tilde{R}_1^{eq}$ and $R_f^{eq}$. Calculate the return on the market portfolio of risky assets in equilibrium.

4. Consider an economy with a single good (e.g., "soybeans") and equally numerous consumers of types *i* and *j*. Consumers have no current endowments and derive no utility from current consumption. Consumers are characterized by the utility functions $u_i = \Sigma_s \ln(c_{is})$ and $u_j = \Sigma_s (c_{js})^{0.5}$, respectively. Assume that there are three states, and that the (future) endowment vectors of the good are $\omega_i$ = (15, 15, 15) and $\omega_j$ = (5, 22.5, 105) for consumers of types *i* and *j*, respectively.
   a. Assume that there is a complete set of Arrow securities. Find the equilibrium prices, consumption plans, and portfolio holdings for this economy. (Note: Normalize prices so that the price of the Arrow security for state 3 equals 1).
   b. Suppose that the same endowment positions are expressed in terms of the holdings of two different assets *A* and *B*. That is, consumers of type *i* (*j*) hold 15 units (1 unit) of asset *A* (*B*) with state returns (1, 1, 1) [(5, 22.5, 105)]. Verify that the complete-market equilibrium cannot be attained if consumers can exchange only assets *A* and *B*.

5. Hansen-Jagannathan Stochastic Discount factor (SDF) Volatility Bounds: To solve the present problem, you should refer to Section 8.1 in Campbell, Lo, and MacKinlay. You should use the data in files hw4data1.txt and hw4data2.txt on the Econ 653 webpage (*http://www.econ.iastate.edu/classes/econ653/lence/*). File hw4data1.txt contains the nominal net rates of return for stocks (S&P 500), 3-month T-bills, and long-term (10-year) government bonds, along with the inflation rate for the U.S. File hw4data2.txt has data on U.S. total population and real aggregate consumption. Please submit your responses as spreadsheets attached to e-mails.
   a. Use the stock and long-term bond data (appropriately deflated by the consumer price index) to calculate the mean-standard deviation combinations $\{\bar{M}, \sigma[M^*(\bar{M})]\}$ for the pricing kernel $M^*(\bar{M})$. Calculate $\sigma[M^*(\bar{M})]$ for $\bar{M} \in [0.9, 1.1]$.
   b. Calculate $SDF_{t+1} = \delta (c_{t+1}/c_t)^{-\gamma}$ using the sum of aggregate consumption on nondurables and services divided by the total population. Use $\delta$ = 0.99 and $\gamma \in [0.5, 30]$.
   c. Plot the $\{\bar{M}, \sigma[M^*(\bar{M})]\}$ values obtained in Problem 5.a and the mean and standard deviations of $SDF_{t+1}$ calculated in Problem 5.b to obtain a graph similar to Figure 8.3 (p. 303) in Campbell, Lo, and MacKinlay. For what values of $\gamma$ is $SDF_{t+1}$ feasible? For what values of $\gamma$ is $SDF_{t+1}$ infeasible? Are the feasible $\gamma$ values "reasonable"? Why or why not? What can you conclude about the consumption CAPM?

**References:**

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay. *The Econometrics of Financial Markets*. Princeton, NJ: Princeton University Press, 1997.