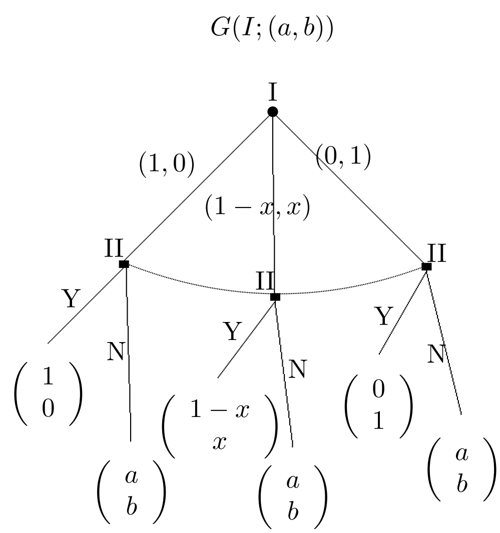
Strategic Bargaining

1 An ultimatum game

Consider the following game, denoted $G(I,(a,b))$. There are two players. Player I first proposes a division $(1-x,x)$ of a pie of size 1. Then, after having heard the proposal, player II responds either by accepting or rejecting the offer. If player II accepts the offer, the proposed division is implemented. If player II rejects the offer, player I gets a and player II gets b, where $a \geq 0$, $b \geq 0$ and $a+b<1$.



Claim 1 The unique subgame perfect equilibrium (SPE) of this game is the following:

Player I proposes the division $(1-b;b)$ and

Player II accepts proposal $(1-x,x)$ if and only if $x \geq b$.

Proof : It is easy to prove that the above pair of strategies is indeed a SPE.

Consider player I's decision node. According to the proposed strategy, he proposes $(1-b, b)$ and player II accepts. If he deviates and proposes $(1-x; x)$ with $x > b$, player II will accept and player I will get $1-x$ which is less than $1-b$. Hence, the deviation is not profitable. If he proposes $(1-x, x)$ with $x < b$, player II will reject the offer, after which the outcome will be (a, b). Since $a + b < 1$, we have that $a < 1 - b$ and therefore the deviation is not profitable.
Consider a node where player II must respond to a proposal $(1-x, x)$.

Case 1: $x \geq b$. In this case, according to the proposed strategy, player II accepts, thereby getting x. If he deviates and rejects the offer, he will get only b. Since $x \geq b$, the deviation is not profitable.

Case 2: $x < b$. In this case, according to the proposed strategy player II rejects and gets b. If he deviates and accepts the offer, he gets x. Since $x < b$, the deviation is not profitable.

This shows that the proposed strategy pair is a SPE. We must now show that there is no other one.

Step 1: (i) In a SPE, at a subgame after I has proposed $(1-x, x)$ with $x > b$, player II must accept. Otherwise, he will end up with b which is less than x.

(ii) At a subgame after I has proposed $(1-x, x)$ with $x < b$, player II must reject and get b.

Step 2: (i) At the root, player 1 cannot propose $(1-x, x)$ with $x > b$. This is because by Step 1 player II will accept and the corresponding payoff for I is $1-x$. But by step 1 player II will also accept a proposal $(1-x+\epsilon, x-\epsilon)$ where $x - \epsilon > b$, thereby getting $1 - x + \epsilon > 1 - x$.

(ii) At the root player 1 cannot propose $(1-x, x)$ with $x < b$. If he makes such an offer, by step 1 player II will reject and I will end up with a. If he proposes $(1-x, x)$ with $x > b$, with x close enough to b, by step 1 player II will accept. Player I's payoff will be very lose to $(1-b)$ which is greater than a.

Step 3: It follows from step 2 that in any SPE player I must propose $(1-b, b)$. But this proposal is a best response only if it is accepted. (If it were rejected, it would be better to propose $a+\epsilon, 1-a-\epsilon$). $\square$

Analogously, we have that the unique SPE of $G(II;(a,b))$ is

Player II proposes $(a, 1-a)$

Player I accepts $(x, 1-x)$ if and only if $x \geq a$. Consequently, the unique SPE outcome is $(a, 1-a)$

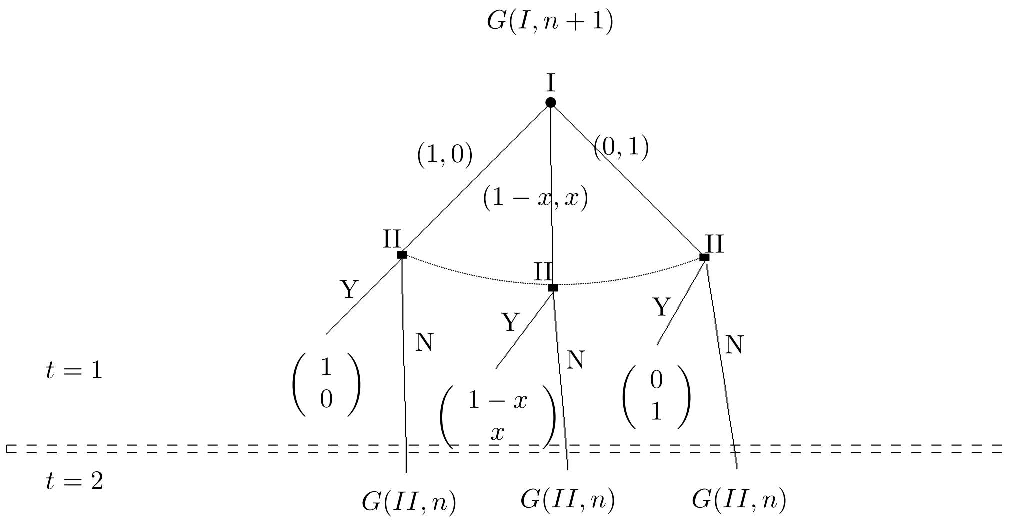
2 An n-stage bargaining game

We are intersted in defining a bargaining game of alternating offers, in which the two players take turns in making proposals. We use an inductive definition.

The game $G(I,1)$ is the ultimatum game where the proposer is player I. In other words, $G(I,1) = G(I,(0,0))$. Analogously the game $G(II,1)$ is defined to be $G(II,(0,0))$.

Assume now that the game $G(II,n)$ has already been defined, for some $n \in \mathbb{N}$. The game of alternating offers $G(I, n+1)$ where Player I is the first proposer and which lasts for at most $n+1$ periods is defined as follows.

At period $t = 1$, player I proposes a division $(1-x, x)$ of the pie, after which player II must respond either by accepting the offer or by rejecting it. If player II accepts the offer, the proposed division is implemented. If player II rejects the proposal, then we enter period 2, at which the players play the game $G(II;n)$, which by assumption has already been defined. The game $G(I, n+1)$ is depicted as follows:



In order to complete the description of the game, we must tell how the players evaluate the possible physical outcomes.

A typical outcome of $G(I, n+1)$ is $\langle (1-x, x); t \rangle$ where $t = 1, \dots, n+1$. The pair $(1-x; x)$ represents the division of the pie and t is the period at which it is implemented. The disagreement outcome is $\langle (0,0); n+1 \rangle$

We assume that the v-N-M utility functions of players I and II are given, respectively, by

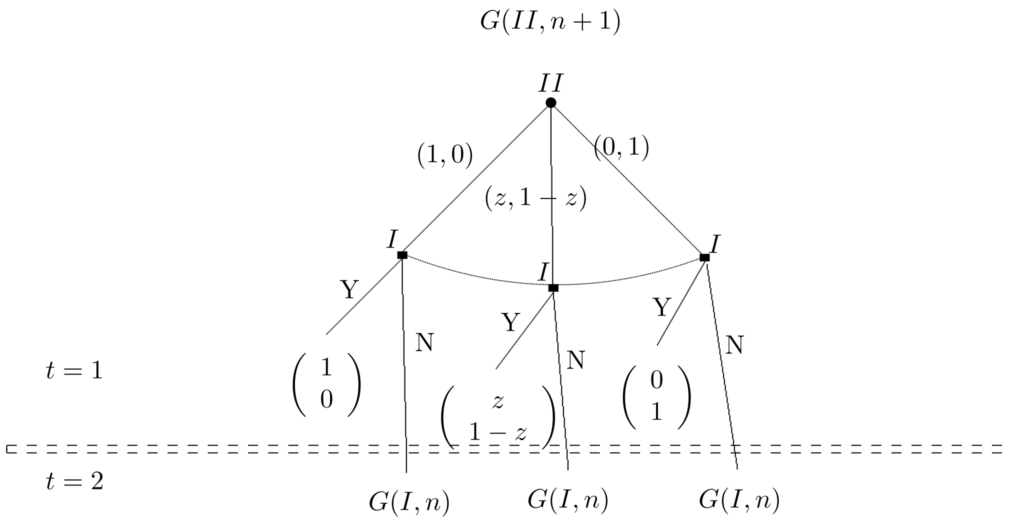
$$u_1((y,x),t) = y\delta_1^{t-1} \qquad 0 < \delta_1 < 1$$

and

$$u_2((y,x),t) = x\delta_2^{t-1} \qquad 0 < \delta_2 < 1.$$

That is the players not only prefer more money to less, but they prefer to get the money sooner rather than later.

Analogously, we can define the game $G(II, n+1)$ which differs from $G(I, n+1)$ only in the fact that player II is the first proposer and that in period 2, after a rejection by player I, they play $G(I, n)$.



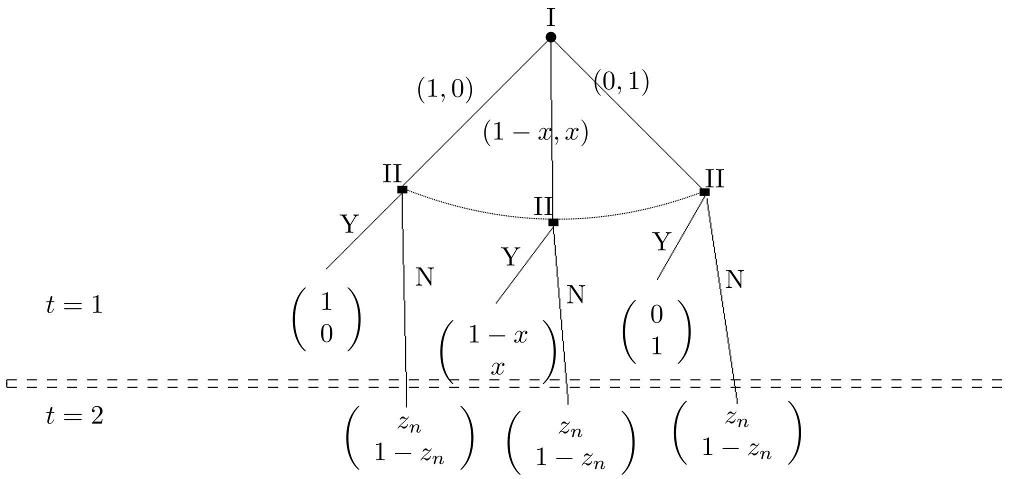
Consider $G(I, 1)$. Since by definition $G(I;1) = G(I;(0,0))$, by Claim 1 the unique SPE outcome is the division $(1, 0)$ proposed and accepted immediately.

Consider $G(II, 1)$. Since $G(II;1) \equiv G(II;(0,0))$, by Claim 1 the unique SPE outcome is: player II proposes $(0, 1)$ and player I accepts.

Assume that $G(II, n)$ has an unique SPE where Player II proposes a division $(z_n, 1 - z_n)$ which is accepted immediately. We want to show that $G(I, n+1)$ has an unique SPE too.

Pick a SPE of $G(I; n+1)$ and denote it s^*. By the definition of SPE, it induces a Nash equilibrium at every one of its subgames. In particular, it induces a Nash equilibrium at $G(II, n)$ and at its subgames, which means that s^* induces a SPE in $G(II, n)$. By the induction hypothesis, $G(II; n)$ has a unique SPE whose corresponding division is $(z_n, 1 - z_n)$. Therefore s^* induces *the* unique SPE of $G(II; n)$ in $G(II; n+1)$.

We can replace the subgame $G(II; n)$ by its unique SPE payoff:



The SPE outcome of $G(II;n)$ is $(z_n; 1 - z_n)$ but at period 2. Therefore, in terms of utilities it is $(\delta_1 z_n, \delta_2(1 - z_n))$

Notice that $\delta_1 z_n \geq 0$, $\delta_2(1 - z_n) \geq 0$ and $\delta_1 z_n + \delta_2(1 - z_n) < 1$. Hence, the above game is just $G(I; (\delta_1 z_n, \delta_2(1 - z_n))$. By Claim 1, this game has a unique SPE which is

Player I proposes $(1 - \delta_2(1 - z_n); \delta_2(1 - z_n))$.

Player II accepts a proposal $(1 - x, x)$ if and only if $x \geq \delta_2(1 - z_n)$.

Therefore, $G(I; n + 1)$ has a unique SPE which is

Player I proposes $\langle 1 - \delta_2(1 - z_n), \delta_2(1 - z_n) \rangle$ at the root and he behaves according to the unique SPE of $G(II;n)$ in these subgames.

Player II accepts an offer $(1 - x, x)$ if and only if $x \geq \delta_2(1 - z_n)$ and he behaves according to the unique SPE of $G(II;n)$ there.

Assuming that $G(I, n)$ has a unique SPE, whose corresponding unique equilibrium division is $(1 - x_n, x_n)$, it can be proved in an analogous way, that $G(II; n + 1)$ has a unique SPE which is:

Player II proposes $\langle \delta_1(1-x_n), 1-\delta_1(1-x_n)\rangle$ at the root and he behaves according to the unique SPE of $G(I;n)$ in these subgames.

Player I accepts an offer $(z, 1-z)$ if and only if $z \geq \delta_1(1-x_n)$ and he behaves according to the unique SPE of $G(I;n)$ there.