1 Algebras and σ-algebras

1.1 Algebras

Definition 1 let X be a space. A collection $\mathcal{A}$ of subsets of X is an *algebra* on X, if and only if it satisfies all the following properties:

1. $X, \emptyset \in \mathcal{A}$
2. For all $A \in \mathcal{A}$, $A^c \in \mathcal{A}$
3. For all $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$

Examples

1. The collection $\mathcal{P}(X)$ of all subsets of X is an algebra,
2. The collection $\mathcal{A} = \{X, \emptyset\}$ is an algebra.
3. Let $X = \{1, 2, 3, 4\}$. The collection $\mathcal{A} = \{\{1, 2, 3, 4\}, \emptyset, \{1\}, \{2, 3, 4\}\}$ is an algebra.

Proposition 1 Let X be a space, $\mathcal{A}$ an algebra on X, and A, B two subsets of X that belong to $\mathcal{A}$. Then,

1. $A \cap B \in \mathcal{A}$
2. $A \setminus B \in \mathcal{A}$
3. $A \Delta B \in \mathcal{A}$

Proof :

1. Since $A, B \in \mathcal{A}$, and $\mathcal{A}$ is an algebra, $(A^c \cup B^c)^c \in \mathcal{A}$. The result follows from the fact that $(A^c \cup B^c)^c = A \cap B$.
2. It follows from the previous item and the fact that $A \setminus B = A \cap B^c$.
3. It follows from the previous item and the fact that $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

$\square$

Proposition 2 Let X be a space, $\mathcal{A}$ an algebra on X. If $A_1, A_2, \dots A_n \in \mathcal{A}$, then

1. $\cup_{i=1}^n A_i \in \mathcal{A}$
2. $\cap_{i=1}^n A_i \in \mathcal{A}$

Proof : Apply induction. $\square$

Example Let $X = \{1, 2, 3, 4, 5\}$, and let $\mathcal{M} = \{\{1\}, \{2\}\}$ be a collection of subsets of X. The follwing collections are algebras that contain $\mathcal{M}$.

- $\mathcal{P}(X)$.
- $\mathcal{A}_1 = \{\{1,2,3,4,5\}, \emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}\}$.
- $\mathcal{A}_2 = \left\{ \begin{array}{l} \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{4,5\}, \{1,2,3\}, \{3,4,5\}, \\ \{1,4,5\}, \{2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,4,5\}, \{1,2,3,4,5\}\} \end{array} \right\}$.

Proposition 3 Let X be a space, and let $(\mathcal{A}_\alpha)_{\alpha \in I}$ be a nonempty collection of algebras on X. Then the collection $\cap_{\alpha \in I} \mathcal{A}_\alpha$ is an algebra on X.

Proof :

1. Since $X, \emptyset \in \mathcal{A}_\alpha$ for all $\alpha \in I$, $X, \emptyset \in \cap_{\alpha \in I} \mathcal{A}_\alpha$
2. Let $A \in \cap_{\alpha \in I} \mathcal{A}_\alpha$. Then $A \in \mathcal{A}_\alpha$ for all $\alpha \in I$. Since $\mathcal{A}_\alpha$ is an algebra for all α, $A^c \in \mathcal{A}_\alpha$ for all $\alpha \in I$. Therefore, $A^c \in \cap_{\alpha \in I} \mathcal{A}_\alpha$.
3. Let $A, B \in \cap_{\alpha \in I} \mathcal{A}_\alpha$. Then $A, B \in \mathcal{A}_\alpha$ for all $\alpha \in I$. Since $\mathcal{A}_\alpha$ is an algebra for all α, $A \cup B \in \mathcal{A}_\alpha$ for all $\alpha \in I$. Therefore, $A \cup B \in \cap_{\alpha \in I} \mathcal{A}_\alpha$.

□

Lemma 1 Let X be a space and let $\mathcal{M}$ be a collection of subsets of X. Let $\mathcal{F}$ be the collection of all the algebras on X that contain $\mathcal{M}$. Then,

1. $\mathcal{F} \neq \emptyset$
2. The intersection of all the elements of $\mathcal{F}$ is an algebra that contains $\mathcal{M}$.

Proof :

1. $\mathcal{P}(X) \in \mathcal{F}$ since $\mathcal{P}(X)$ is an algebra that contains $\mathcal{M}$.
2. By Proposition 3, the intersection of all the algebras in $\mathcal{F}$ is an algebra. Since each one of them contains $\mathcal{M}$, so does the intersection.

□

Definition 2 Let X be a space, and let $\mathcal{M}$ be a collection of subsets of X. The intersection of all the algebras on X that containt $\mathcal{M}$ is called *the algebra generated by* $\mathcal{M}$, and it is denoted by $\mathcal{A}(\mathcal{M})$.

Theorem 1 Let X be a space, $\mathcal{M}$ a collection of subsets of X, and $\mathcal{A}$ algebra that contains $\mathcal{M}$. Then $\mathcal{A} = \mathcal{A}(\mathcal{M})$ if and only if $\mathcal{A}$ is contained in every algebra that contains $\mathcal{M}$.

Proof : Assume $\mathcal{A} = \mathcal{A}(\mathcal{M})$. Since by definition $\mathcal{A}(\mathcal{M})$ is contained in every algebra that contains $\mathcal{M}$, so does $\mathcal{A}$. Assume now that $\mathcal{A}$ is contained in every algebra that contains $\mathcal{M}$. In particular, it is contained in $\mathcal{A}(\mathcal{M})$. That is, $\mathcal{A} \subseteq \mathcal{A}(\mathcal{M})$. But since $\mathcal{A}$ is an algebra that contains $\mathcal{M}$, by definition of $\mathcal{A}(\mathcal{M})$, $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{A}$. □

Lemma 2 Let $\mathcal{M}$ and $\mathcal{K}$ be two collections of subsets of a space X. If $\mathcal{M} \subseteq \mathcal{K} \subseteq \mathcal{A}(\mathcal{M})$, then $\mathcal{A}(\mathcal{M}) = \mathcal{A}(\mathcal{K})$.

Proof : Since $\mathcal{A}(\mathcal{M})$ is an algebra that contains $\mathcal{K}$, by definition of $\mathcal{A}(\mathcal{K})$, we have

$$\mathcal{A}(\mathcal{K}) \subseteq \mathcal{A}(\mathcal{M}).$$

Therefore, since $\mathcal{M} \subseteq \mathcal{K} \subseteq \mathcal{A}(\mathcal{K})$, $\mathcal{A}(\mathcal{K})$ is an algebra that contains $\mathcal{M}$ and that is contained in every algebra that contains $\mathcal{M}$. By Theorem 1, $\mathcal{A}(\mathcal{M}) = \mathcal{A}(\mathcal{K})$. □

1.2 σ-Algebras

Definition 3 let X be a space. A collection $\mathcal{A}$ of subsets of X is a *σ-algebra* on X, if and only if it satisfies all the following properties:

1. $X, \emptyset \in \mathcal{A}$
2. For all $A \in \mathcal{A}$, $A^c \in \mathcal{A}$
3. For all sequence $(A_n)_{n=1}^{\infty}$ of elements of $\mathcal{A}$, $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$.

It should be clear that every σ-algebra is an algebra, since every union of two sets A and B can be written as a countable union $(A \cup B \cup \emptyset \cup \emptyset \ldots)$. However, not all algebras are σ-algebras.
Example:

Let $X = \{1, 2, 3, \ldots\}$, and let $\mathcal{A} = \{A \subseteq X : A \text{ is finite or } A^c \text{ is finite}\}$. It is easy to see that $\mathcal{A}$ is an algebra on X. Since $\emptyset$ is finite, both X and $\emptyset$ are in $\mathcal{A}$. Let $A, B \in \mathcal{A}$. If both A and B are finite, then so is $A \cup B$. Otherwise, at least one of them is not finite, but its complement is. Assume that A is such set. Then $(A \cup B)^c = A^c \cap B^c \subseteq A^c$ which is finite. So $(A \cup B)^c$ is finite and therefore is in $\mathcal{A}$. Lastly, since $(A^c)^c = A$, if $A \in \mathcal{A}$, so is A^c. On the other hand, $\mathcal{A}$ is not a σ-algebra. Indeed, for all $n = 1, 2, \ldots$, the set $\{2n\}$ is in $\mathcal{A}$, but $\cup_{n=1}^{\infty} \{2n\} \notin \mathcal{A}$.