Second Price Auction

The bayesian game is given by $\langle N, \Omega, (A_i, \mu_i, \mathcal{P}_i, u_i)_{i \in N} \rangle$ where

- $N = \{1, \dots, n\}$
- $\Omega = [0, \overline{v}]^n = \{(v_1, \dots, v_n) : 0 \le v_i \le \overline{v}\}$
- $A_i = [0, \infty)$
- $\mathcal{P}_i(\hat{v}_1, \dots, \hat{v}_n) = \{(v_1, \dots, v_n) \in \Omega : v_i = \hat{v}_i\}$
- μ_i: any.
- $u_i(a, (v_1, \dots, v_n)) = \begin{cases} \frac{v_i - \max\{a_j : j \ne i\}}{m} & \text{if } a_i \ge \max\{a_j : j \ne i\} \text{ and } m = |\{j : a_j = a_i\}| \\ 0 & \text{otherwise} \end{cases}$

The action $a_i = v_i$ is an optimal action for agent i of type v_i, independently of the other players' actions. If $\max\{a_j : j \ne i\} \ge v_i$, then the maximum utility is 0 and can be achieved by setting $a_i = v_i$. If $\max\{a_j : j \ne i\} < v_i$, then the maximum utility is $\frac{v_i - \max\{a_j : j \ne i\}}{m}$ and can be achieved by setting $a_i = v_i$.

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- $\mathcal{P}_i(\hat{v}_1, \dots, \hat{v}_n) = \{(v_1, \dots, v_n) \in \Omega : v_i = \hat{v}_i\}$
- $\mu_i((v_1, \dots, v_n) \le ((\hat{v}_1, \dots, \hat{v}_n)) = F(\hat{v}_1) \times \dots \times F(\hat{v}_n)$
- $u_i(a, (v_1, \dots, v_n)) = \begin{cases} \frac{v_i - a_i}{m} & \text{if } a_i \ge \max\{a_j : j \ne i\} \text{ and } m = |\{j : a_j = a_i\}| \\ 0 & \text{otherwise} \end{cases}$

We guess that there is a symmetric equilibrium in which a bidder with valuation v bids the price $\beta(v)$. Our objective is to find the function β that tells us the price offered by a buyer of type v. We also guess that the function β is strictly increasing.

So assume that all $n-1$ bidders adopt strategy β. What is the best response of a bidder whose valuation is v? If he bids b then either he bid the highest bid and gets the object, or he does not bid the highest bid and he does not get object. The probability that bid b is the highest is:

$$\begin{aligned}
\text{Prob}[\max\{\beta(v_j) : j \ne i\} \le b] &= \text{Prob}[\beta(v_1) \le b] \dots \text{Prob}[\beta(v_{i-1}) \le b]\text{Prob}[\beta(v_{i+1}) \le b] \dots \text{Prob}[\beta(v_n) \le b] \\
&= \text{Prob}[v_1 \le \beta^{-1}(b)] \dots \text{Prob}[v_{i-1} \le \beta^{-1}(b)]\text{Prob}[v_{i+1} \le \beta^{-1}(b)] \dots \text{Prob}[v_n \le \beta^{-1}(b)] \\
&= \underbrace{F[\beta^{-1}(b)] \dots F[\beta^{-1}(b)]}_{n-1} \\
&= F^{n-1}[\beta^{-1}(b)]
\end{aligned}$$

where $\beta^{-1}(b)$ is the type that bids b according to strategy β. The probability that b is not the highest bid is $1 - F^{n-1}[\beta^{-1}(b)]$.

Consequently, the expected utility associated with a bid b is

$$\pi(b, v) = (v - b)F^{n-1}[\beta^{-1}(b)]. \tag{1}$$

The buyer will choose b so as to maximize the above functon. The necessary contidion for b to be a best response is:

$$\frac{\partial \pi}{\partial b} = 0.$$

If β is an equilibrium strategy we must have

$$b^* = \beta(v).$$

Let's define now the indirect utility function:

$$V(v) \equiv \pi(\beta(v), v) = (v - \beta(v))F^{n-1}(v).$$

If I knew $V(v)$, then I could easily deduce $\beta(v)$:

$$\beta(v) = v - \frac{V(v)}{F^{n-1}(v)}.$$

The problem is that I do not know V. Let's try to figure it out.

$$\frac{dV}{dv} = \frac{\partial \pi}{\partial b}\frac{\partial \beta}{\partial v} + \frac{\partial \pi}{\partial v} = F^{n-1}(v).$$

Therefore,

$$\begin{aligned} V(v) - V(0) &= \int_0^v \frac{dV}{dv}(x)dx \\ &= \int_0^v F^{n-1}(x)dx. \end{aligned}$$

Consequently,

$$\begin{aligned} V(v) &= V(0) + \int_0^v F^{n-1}(x)dx \\ &= \int_0^v F^{n-1}(x)dx. \end{aligned}$$

As a result,

$$\beta(v) = v - \frac{\int_0^v F^{n-1}(x)dx}{F^{n-1}(v)}.$$

Taking derivatives of β with respect to v we can check that β is indeed increasing.