1 Preliminaries

Let I be a finite set of agents. Let (Ω, Σ, μ) be a probability space where

- Ω is a countable set of states,
- Σ is a σ-algebra of *events* and
- μ is a probability measure on Σ.

For each $i \in I$, Π_i is a partition of Ω into measurable sets of positive measure. For $\omega \in \Omega$, $\Pi_i(\omega)$ denotes de element of the partition Π_i that contains ω.

We denote by $\mathcal{F}_i$ the σ-algebra generated by Π_i.

Claim 1 $\mathcal{F}_i$ is the set of all unions of elements of Π_i

Proof: Denote by $\mathcal{M}$ the set of all unions of elements of Π_i. The σ-algebra generated by Π_i is the intersection of all the σ-algebras that contain Π_i. But every σ-algebra that contains Π_i, must contain all the (countable) unions of elements of Π_i. So, $\mathcal{M} \subseteq \mathcal{F}_i$. In order to show that $\mathcal{F}_i \subseteq \mathcal{M}$ it is enough to show that the set of unions of elements of Π_i, $\mathcal{M}$, is a σ-algebra that contains Π. To see this, first note that $\Omega \in \mathcal{M}$ since Ω is the union of all the elements of Π_i. Also $\emptyset \in \mathcal{M}$ since $\emptyset$ is the empty union. Second, if $A, B \in \mathcal{M}$ then both A and B are unions of elements of Π_i. Therefore so is their union. Lastly, let $A \in \mathcal{M}$. Then, there is a set of indices I such that $A = \cup_{\alpha \in I} P_\alpha$ where $P_\alpha \in \Pi_i$ for all $\alpha \in I$. We need to show that $A^c \in \mathcal{M}$.

$$\begin{aligned} A^c &= (\cup_{\alpha \in I} P_\alpha)^c \\ &= \cap_{\alpha \in I} (P_\alpha)^c \\ &= \cap_{\alpha \in I} (\Omega \setminus P_\alpha) \\ &= \Omega \setminus \cup_{\alpha \in I} P_\alpha \\ &= \cup_{\alpha \notin I} P_\alpha. \end{aligned}$$

QED

Definition 1 We say that i *knows event* E *at* ω if $\Pi_i(\omega) \subseteq E$.

Claim 2 Agent i knows event E at ω if and only if there exists $A \in \mathcal{F}_i$ such that $\omega \in A \subseteq E$

Proof : $\omega \in K_i(E)$ implies that $\Pi_i(\omega) \subseteq E$. Since $\omega \in \Pi_i(\omega) \in \mathcal{F}_i$, $\Pi_i(\omega)$ is the set A we are looking for. Assume now that $\omega \in A \subseteq E$ for some $A \in \mathcal{F}_i$. Since $A \in \mathcal{F}_i$, A is a union of elements of Π_i. Since $\omega \in A$, we must have $\Pi_i(\omega) \subseteq A$. Therefore, $\Pi_i(\omega) \subseteq E$, and $\omega \in K_i(E)$. □

Definition 2 Let $K_i(E)$ be the event "i knows E." That is

$$K_i(E) = \{\omega \in \Omega : \Pi_i(\omega) \subseteq E\}.$$

Claim 3 Let E be an event, and let $\mathcal{M} = \{A \in \mathcal{F}_i : A \subseteq E\}$ be the set of $\mathcal{F}_i$-measurable subsets of E. Then, $K_i(E) = \cup_{A \in \mathcal{M}} A$.

Proof : Let $\omega \in K_i(E)$. Then, by Claim 2, there exists $A \in \mathcal{F}_i$ such that $\omega \in A \subseteq E$. This means that $\omega \in A \in \mathcal{M}$ and consequently, $\omega \in \cup_{A \in \mathcal{M}} A$. Conversely, assume $\omega \in \cup_{A \in \mathcal{M}} A$. Then there is $A \in \mathcal{F}_i$ such that $\omega \in A \subseteq E$, which, by Claim 2 means that $\omega \in K_i(E)$. □

Claim 4 $K_i(E) \in \mathcal{F}_i$.

Proof : By, Claim 3, $K_i(E)$ is a union of elements of $\mathcal{F}_i$. Since $\mathcal{F}_i$ is a σ-algebra, $K_i(E)$ belongs to $\mathcal{F}_i$. □

Claim 5 $K_i(E) \subseteq E$.

Proof : By, Claim 3, $K_i(E)$ is a union of of subsets of E. As such, $K_i(E)$ itself is a subset of E. □

Claim 6 For all events $E \in \Sigma$, $K_i(E) = E$ if and only if $E \in \mathcal{F}_i$.

Proof : If $K_i(E) = E$, then $E \in \mathcal{F}_i$ by Claim 4. Conversely, if $E \in \mathcal{F}_i$, Then, since $E \subseteq E$, $E \in \mathcal{M}$ and by Claim 3 $E \subseteq K_i(E)$. Since by Claim 5 $K_i(E) \subseteq E$, we have that $K_i(E) = E$. □

Claim 7 For all sequences of events $(E_n)_{n \geq 1}$, $K_i(\cap_{n \geq 1} E_n) = \cap_{n \geq 1} K_i(E_n)$

Proof : By Claim 3

$$\begin{aligned} K_i\Big(\bigcap_{n \geq 1} E_n\Big) &= \bigcup_{\substack{A \in \mathcal{F}_i \\ A \subseteq \cap_{n \geq 1} E_n}} A \\ &= \bigcap_{n \geq 1} \bigcup_{\substack{A \in \mathcal{F}_i \\ A \subseteq E_n}} A \\ &= \bigcap_{n \geq 1} K_i(E_n). \end{aligned}$$

□

Claim 8 For all $E, F \in \Sigma$, if $E \subseteq F$ then $K_i(E) \subseteq K_i(F)$.

Proof : Since $K_i(E) \subseteq E \subseteq F$, $K_i(E)$ is an element of $\mathcal{F}_i$ that is contained in F. By Claim 3, $K_i(E) \subseteq K_i(F)$. □

Definition 3 An event E is *common knowledge* at $\omega \in \Omega$ if there exists an event $A \in \mathcal{F}_i$ for all $i \in I$ such that $\omega \in A \subseteq E$.

Let E be an event. Define recursively the following events:

$$\begin{aligned} E^0 &= E \\ E^n &= \cap_{i \in N} K_i(E^{n-1}) \end{aligned}$$

and $C(E) = \cap_{n \geq 1} E^n$.

Proposition 1 (Monderer and Samet (1989)) E is common knowledge at ω if and only if $\omega \in C(E)$.

Proof : Assume that E is common knowledge at ω. Then, there is an event $A \in \mathcal{F}_i$ for all $i \in N$ such that $\omega \in A \subseteq E$. It is enough to show that $A \subseteq C(E)$. Since $A \in \mathcal{F}_i$ for all $i \in N$ and $A \subseteq E$, by Claims 8 and 6,

$$A = K_i(A) \subseteq K_i(E) \qquad \forall i \in N$$

which implies

$$A \subseteq \cap_{i \in N} K_i(E) = E^1.$$

Assume that for $n \geq 1$, $A \subseteq E^n$. Then again by Claims 8 and 6,

$$A = K_i(A) \subseteq K_i(E^n) \qquad \forall i \in N$$

which implies

$$A \subseteq \cap_{i \in N} K_i(E^n) = E^{n+1}.$$

We have proved inductively that $A \subseteq E^n$ for all $n \geq 1$. Therefore $A \subseteq \cap_{n \geq 1} E^n = C(E)$.

Assume now that $\omega \in C(E)$. We need to show that there is an event $A \in \mathcal{F}_i$ for all $i \in N$ such that $\omega \in A \subseteq E$. We will show that $C(E)$ is the event we are looking for. To see that $C(E) \subseteq E$, note that $C(E) \subseteq E^1 \subseteq K_i(E) \subseteq E$. To see that $C(E) \in \mathcal{F}_i$ for all $i \in N$, note that for all $n \geq 1$ and for all $i \in N$,

$$C(E) \subseteq E^{n+1} \subseteq K_i(E^n).$$

Therefore,

$$\begin{aligned} C(E) &\subseteq \cap_{n \geq 1} K_i(E^n) \\ &\subseteq K_i(\cap_{n \geq 1} E^n) \\ &\subseteq K_i(C(E)). \end{aligned}$$

The result follows from Claims 5 and 6. □

2 Agreeing to disagree

Definition 4 A function $f : \Omega \to \mathbb{R}$ is *measurable with respect to partition* Π_i, if for all $\omega, \omega' \in \Omega$, $\Pi_i(\omega) = \Pi_i(\omega')$ implies $f(\omega) = f(\omega')$.

Definition 5 Let $f : \Omega \to \mathbb{R}$ be a measurable function. An expectation of f conditional on Π_i, is a Π_i-measurable function $E(f|\Pi)$ such that

$$\sum_{\omega \in A} E(f|\Pi)(\omega)\, \mu(\omega) = \sum_{\omega \in A} f(\omega)\, \mu(\omega) \qquad \text{for all } A \in \mathcal{F}_i.$$

Example 1 Let $\{\Omega, \Sigma, \mu\}$ be a probability space, and let $E \in \Sigma$ be an event. Let Π_i be a partition of Ω into events of positive probability. For each $\omega \in \Omega$, $\Pi_i(\omega)$ is the element of Π that contains ω. Recall that the probability of E conditional on $\Pi_i(\omega)$ is given by

$$\mu(E|\Pi_i(\omega)) = \frac{\mu(E \cap \Pi_i(\omega))}{\mu(\Pi_i(\omega))}.$$

Define the function $\mu(E|\Pi_i) : \Omega \to [0, 1]$ by

$$\mu(E|\Pi_i)(\omega) = \frac{\mu(E \cap \Pi_i(\omega))}{\mu(\Pi_i(\omega))}.$$

$\mu(E|\Pi_i)(\omega)$ is the conditional probability of E given $\Pi_i(\omega)$.

Define the function $f : \Omega \to [0, 1]$ by

$$f(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases}$$

It turns out that the conditional expectation $E(f|\Pi_i)$ equals $\mu(E|\Pi_i)$. To see this we need to show that

$$\sum_{\omega' \in \Pi_i(\omega)} \mu(E|\Pi_i)(\omega')\mu(\omega') = \sum_{\omega' \in \Pi_i(\omega)} f(\omega')\mu(\omega').$$

And indeed,

$$\begin{aligned} \sum_{\omega' \in \Pi_i(\omega)} \mu(E|\Pi_i)(\omega')\mu(\omega') &= \sum_{\omega' \in \Pi_i(\omega)} \frac{\mu(E \cap \Pi_i(\omega'))}{\mu(\Pi_i(\omega'))}\mu(\omega') \\ &= \sum_{\omega' \in \Pi_i(\omega)} \frac{\mu(E \cap \Pi_i(\omega))}{\mu(\Pi_i(\omega))}\mu(\omega') \\ &= \mu(E \cap \Pi_i(\omega)). \end{aligned}$$

and

$$\begin{aligned} \sum_{\omega' \in \Pi_i(\omega)} f(\omega')\mu(\omega') &= \sum_{\omega' \in E \cap \Pi_i(\omega)} \mu(\omega') \\ &= \mu(E \cap \Pi_i(\omega)) \end{aligned}$$

Proposition 2 Let x_i for $i \in I$ be a collection of numbers, one for each agent, and let $C_i = \{\omega \in \Omega : E(f|\Pi_i)(\omega) = x_i\}$ be the event at which the expected value of f conditional on Π_i is x_i. Let $C = \cap_{i \in N} C_i$. If C is common knowledge at ω then $x_i = x_j$ for all $i, j \in I$.

Proof : Assume that C is common knowledge at ω among the members of I. Then, there is an event $A \in \cap_{i \in I} \mathcal{F}_i$ for all $i \in I$ such that $\omega \in A \subseteq C$. On the one hand, since $A \subseteq C$, and $E(f|\Pi_i)(\omega) = x_i$ for all $\omega \in C$, we have that

$$\begin{aligned}\sum_{\omega \in A} E(f|\Pi_i)(\omega)\,\mu(\omega) &= \sum_{\omega \in A} x_i\,\mu(\omega) \\ &= x_i\,\mu(A) \qquad \forall i \in I.\end{aligned}$$

On the other hand, since $A \in \mathcal{F}_i$ for all $i \in I$,

$$\sum_{\omega \in A} E(f|\Pi_i)(\omega)\,\mu(\omega) = \sum_{\omega \in A} f(\omega)\,\mu(\omega) \qquad \forall i \in I.$$

As a result,

$$x_i\,\mu(A) = \sum_{\omega \in A} f(\omega)\,\mu(\omega) \qquad \forall i \in I,$$

which implies that

$$x_i = x_j \quad \text{for all } i, j \in I.$$

□