Economics 618 Fall 2004

Formal Description of Bargaining Situations

Situation 1

There are two individuals and one dollar to be assigned to one of them. They must agree on who gets the dollar. If they do not reach an agreement, no one gets the dollar.

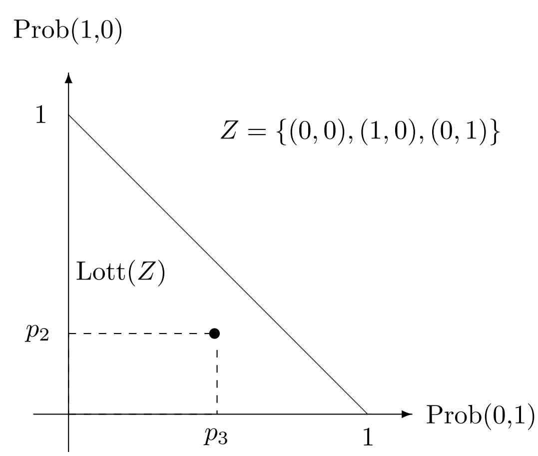
What is the set of possible outcomes? There are three possible physical outcomes.

$$Z = \{(0,0); (1;0); (0,1)\}$$

They may decide to use some random mechanism, or they may face some chance that their negotiations break down. In order to allow for such random outcomes, let's consider the set of lotteries over the set of feasible physical outcomes:

$$\text{Lott}(Z) = \{(p_1, p_2, p_3) : p_i \geq 0,\ \sum_{i=1}^{3} p_i = 1\}.$$

Graphically:



Assume now that the individuals have von Neumann-Morgenstern utilities given by $u_1(x_1, x_2) = \sqrt{x_1}$ and $u_2(x_1, x_2) = \log_2(x_2 + 1)$ respectively. Therefore, the utility assigned by 1 to any lottery $p = (p_1, p_2, p_3)$ is

$$E_p(u_1) = p_1\sqrt{0} + p_2\sqrt{1} + p_3\sqrt{0} = p_2.$$

The utility assigned by 2 for any lottery p is

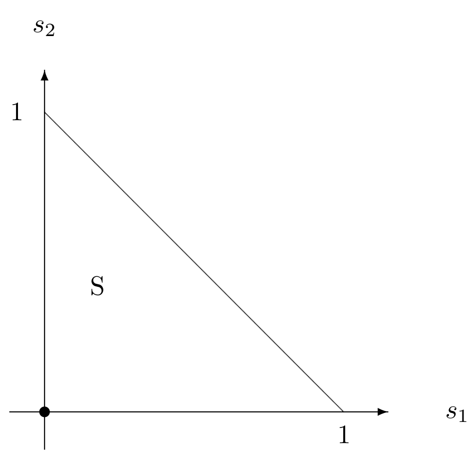
$$E_p(u_2) = p_1 \log_2(1) + p_2 \log_2(1) + p_3 \log_2(2) = p_3.$$

The utility possibility set is the set of all pairs of utilities that can be achieved by some lottery:

$$S = \{(s_1; s_2) : s_1 = E_p(u_1),\ s_2 = E_p(u_2) \text{ for some } p \in \text{Lott}(Z)\}$$

or

$$S = \{(s_1, s_2) : s_1 = p_2,\ s_2 = p_3,\ p_2 \geq 0,\ p_3 \geq 0,\ p_2 + p_3 \leq 1\}$$



We see that this utility possibility set is convex, compact and contains the point (0,0) which is the pair of utilities associated with the disagreement.

Let's apply an affine transformation to the utility of agent 1: $v_1(x_1, x_2) = 2\sqrt{x_1} + 1$. Since u_1 is a von Neumann-Morgenstern utility function, we know that v_1 represent the same preferences as before. What is the new utility possibilities set?

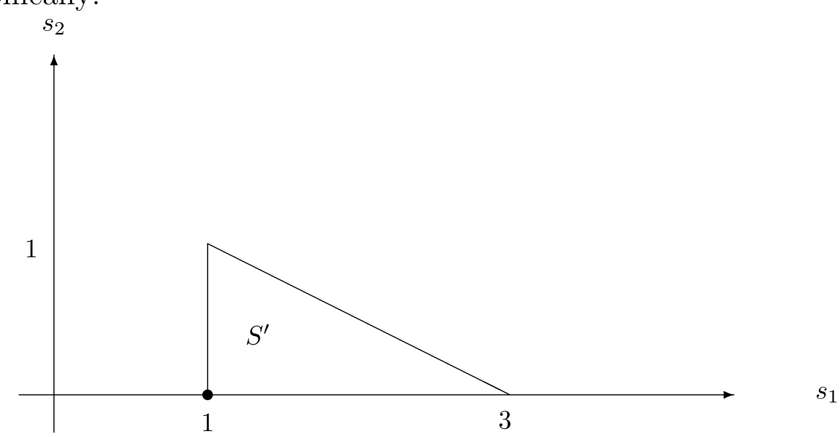
The set of physical outcomes hasn't changed. Consequently, the set of lotteries over Z hasn't changed either. What is the utility of 1 of an arbitrary lottery $p \in \text{Lott}(Z)$?

$$E_p(v_1) = 2E_p(u_1) + 1 = 2p_2 + 1.$$

After the new transformation, the utility possibility set is, by definition

$$S' = \{(s_1, s_2) : s_1 = 2p_2 + 1,\ s_2 = p_3,\ p_2 \geq 0,\ p_3 \geq 0,\ p_2 + p_3 \leq 1\}.$$

Graphically:



Again, we have a set which is convex, compact and contains the point (1;0) corresponding to the utilities associated to the disagreement.

Situation 2: Consider the same two individuals as before, but now we have two dollars to distribute. As before, if they do not reach an agreement, both get 0.

The set of physical outcomes:

$$Z = \{(0;0),(2;0),(1;1),(0;2)\}.$$

Given that they can agree on any random device to distribute the money, let's calculate the set of lotteries over Z

$$\text{Lott}(Z) = \{(p_1;p_2;p_3;p_4) : p_i \geq 0,\ \sum_{i=1}^{4} p_i = 1\}.$$

Assume again that the individuals' utility functions are given by $u_1(x,y) = \sqrt{x}$ and $u_2(x,y) = \log_2(y+1)$, respectively. Then, for any given lottery $p \in \text{Lott}(Z)$, the corresponding expected utilities are given by:

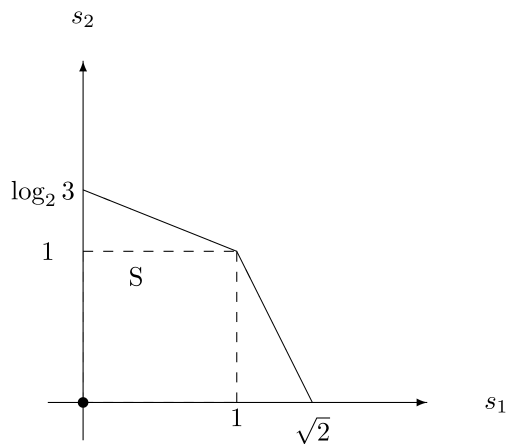
$$\begin{aligned} E_p(u_1) &= p_1\sqrt{0} + p_2\sqrt{2} + p_3\sqrt{1} + p_4\sqrt{0} \\ &= p_2\sqrt{2} + p_3 \end{aligned}$$

and

$$\begin{aligned} E_p(u_2) &= p_2 \log_2(1) + p_2 \log_2(1) + p_3 \log_2(2) + p_4 \log_2 3 \\ &= p_3 + p_4 \log_2 3 \end{aligned}$$

The set of utility possibilities is

$$S = \{(s_1,s_2) : s_1 = p_2\sqrt{2}+p_3,\ s_2 = p_3+p_4\log_2 3 \text{ for some } (p_1,p_2,p_3,p_4) \in \text{Lott}(Z)\}$$



Again we have that the utility possibility set is a compact convex set, that contains the utility pair associated with the disagreement.

Situation 3:

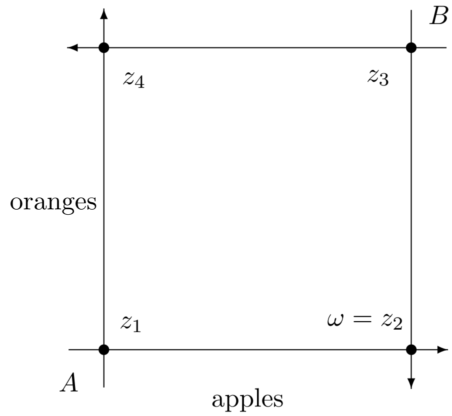
There are two individuals, A and B, and two goods, apples and oranges.

The initial endowment of A is $w_A = (1; 0)$ meaning that he owns 1 apple and no oranges. B's initial endowment is $w_B = (0, 1)$ meaning that he owns 1 orange and no apples.

An allocation is a distribution of the goods between the agents: $\langle (x_1^A, x_2^A), (x_1^B, x_2^B) \rangle$ s.t.

$$x_1^A + x_1^B = 1$$

$$x_2^A + x_2^B = 1$$



There are four possible allocations, which are the outcomes:

$$Z = \{\langle(1,1),(0,0)\rangle, \langle(1,0),(0,1)\rangle, \langle(0,1),(1,0)\rangle, \langle(0,0),(1,1)\rangle\}$$

The agents need to bargain over the outcomes. If they do not reach an agreement, they remain with the initial endowment $= \langle(1,0),(0,1)\rangle$.

The individuals' von Neumann-Morgentern utility functions are given by

$$u_A(x_1, x_2) = (x_1^{1/2} + x_2^{1/2})^2$$
$$u_B(x_1, x_2) = x_1 + x_2$$

The set of lotteries over the allocations is

$$\text{Lott}Z = \{p_1, p_2, p_3, p_4) : p_i \geq 0, \ \sum_{i=1}^{4} p_i = 1\}.$$

The corresponding utilities of the degenerate lotteries are

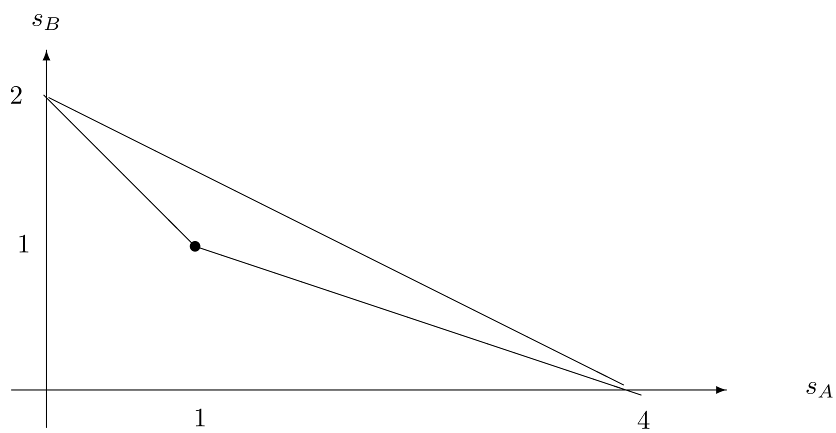
$$u_A(1,1) = 4 \quad u_B(0,0) = 0$$
$$u_A(1,0) = 1 \quad u_B(0,1) = 1$$
$$u_A(0,1) = 1 \quad u_B(1,0) = 1$$
$$u_A(0,0) = 0 \quad u_B(1,1) = 2$$

Therefore, the expected utilities of a lottery $p \in \text{Lott}(Z)$ are, respectively:

$$\begin{aligned} E_p(u_A) &= 4p_1 + 1p_2 + 1p_3 + 0p_4 = 4p_1 + p_2 + p_3 \\ E_p(u_B) &= 0p_1 + p_2 + p_3 + 2p_4 = p_2 + p_3 + 2p_4 \end{aligned}$$

The set of utility possibilities is

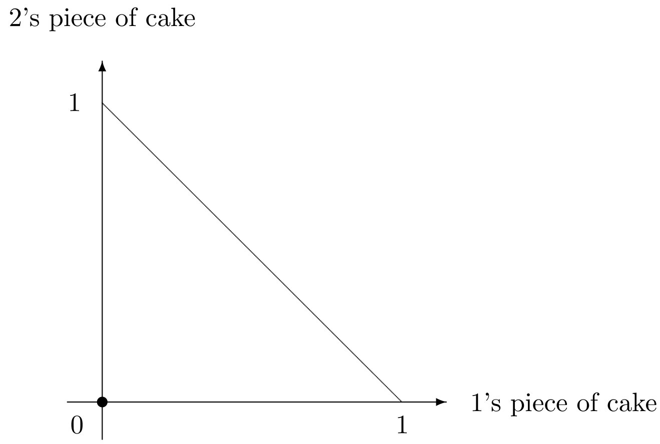
$$S = \{(s_A, s_B) : s_A = p_2+p_3+4p_4,\ s_B = p_2+p_3+2p_1,\ (p_1, p_2, p_3, p_4) \in \text{Lott}(Z)\}$$



Again, we have the result that the utility possibility set is a compact convex set that contains the disagreement point.

Situation 4: Suppose there is one cake to be divided between two agents. It can be divided in any way you want but if there is no agreement both agents get 0. Namely, the set of all possible division i's

$$Z = \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 = 1\} \cup \{0, 0\}.$$



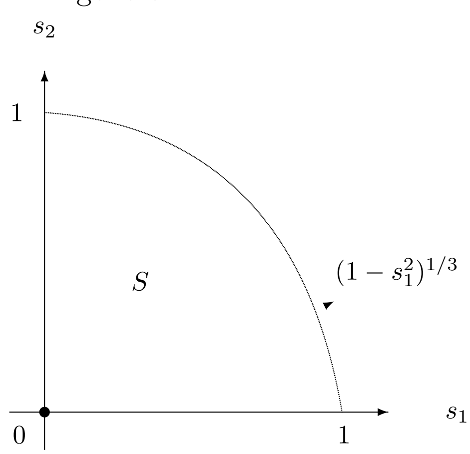
Assume that the utilities of the agents are, respectively:

$$u_1(x_1, x_2) = x_1^{1/2} \text{ and } u_1(x_1, x_2) = x_2^{1/3}.$$

Since $x_1 + x_2 = 1$ for any division of the cake, we have that whenever the cake is divided the utilities of the agents are related by

$$u_2(x_1, x_2) = (1 - x_1)^{1/3} = (1 - u_1^2)^{1/3}.$$

Hence, if the division is $(1;0)$, $u_1 = 1$ and $u_2 = 0$. If the division is $(0, 1)$ then $u_1 = 0$ and $u_2 = 1$. In general:



The set of attainable utilities is therefore the convex hull of the above utility combinations, together with the disagreement point $(0, 0)$.