

Description of a Game by Means of a Tree

The game of chess belongs to the big family of two player games, in which after the first player takes his first move, both players alternate in making moves until the game ends. At the end of each play, a final outcome is determined.

Example 1 There is a board with four squares, each numbered with either 1, 2, 3 or 5. In the game, two players play: Player I and player II. Player I opens and he conquers one of the squares. Afterwards, player II conquers another square, under the following conditions:

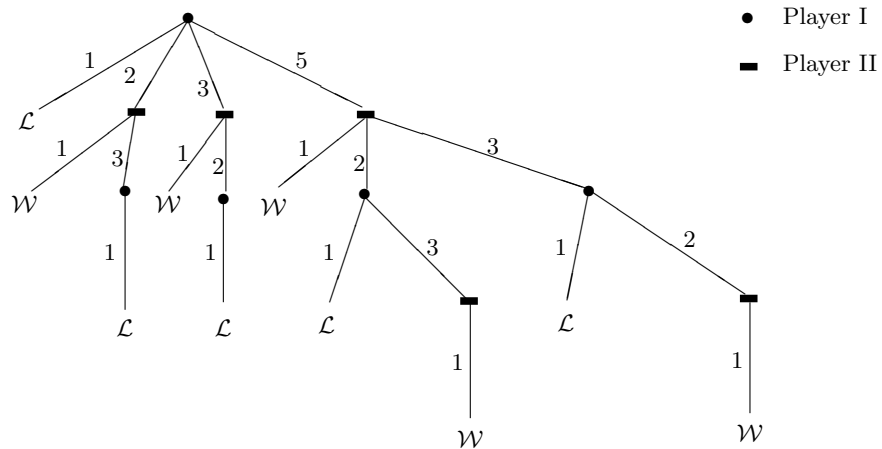
- (a) The square has not been conquered yet by one of the players.
- (b) The square's number is not bigger than 1 plus the number in the square that has been conquered in the previous move.

1	2
3	5

When square number 1 is conquered, the game is finished, the player who conquers this square is the one who loses.

In this game, every play of the game has one of the following two outcomes: either player I wins or player II wins.

We can describe the game in the following way:

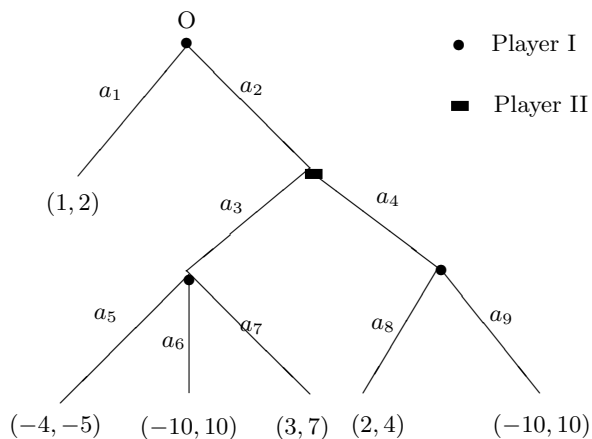


A graphic description of this sort is called the tree of a game. And a game that results from a graphic description of this kind is called an extensive form game.

The beginning of the game is the root of the game. The possible actions of the players are described by edges.

Definition 1 A *play* of the game is a description of all the actions taken, from the beginning of the game until its end and is graphically given by a path.

In order for the description of the game to be complete, we must say what the final outcomes are. The outcomes are physical outcomes and can be for instance player I wins, player II wins, player I pays \$2 to player II, etc.

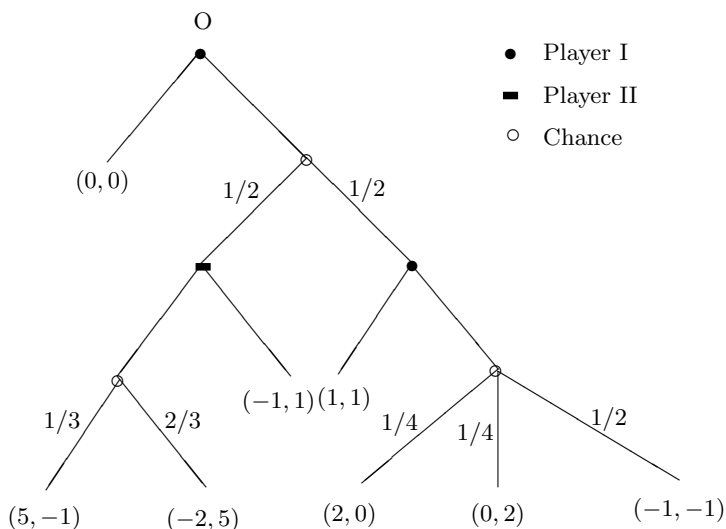


In this game, there are six possible plays of the game. In the previous one, there are 10 plays.

Chance Moves

Up until now we have dealt with games where the possible moves are taken by the players themselves. Now we'll expand the family of games by allowing for chance moves to decide on the moves.

Example 2 The following tree represents a game with chance moves:

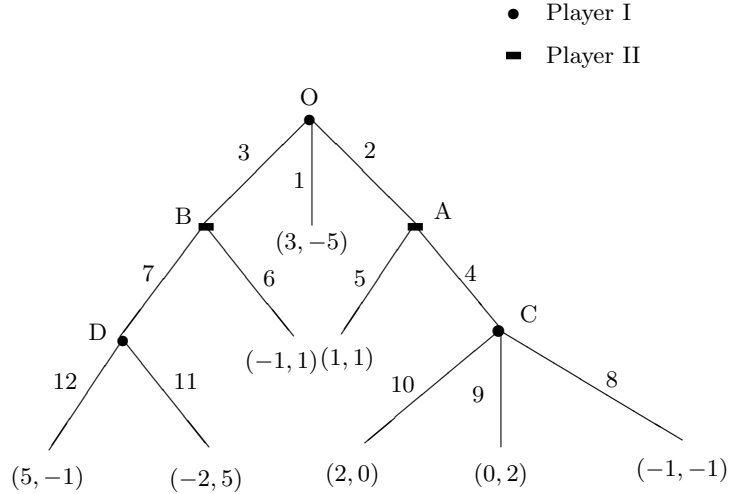


Strategies

A strategy for player i is a statement that specifies an action at *each* of the decision nodes at which it would be player i 's duty to make a decision if that node were actually reached.

If all the players in a game select a strategy and stick with it, then this completely determines how a game without chance moves will be played.

Example 3 Consider the following game in extensive form:



Player I has three decision nodes: O, C, and D. A possible strategy is:

$$s(O) = 2 \quad s(C) = 9 \quad s(D) = 12$$

Player II has two decision nodes: A and B. A possible strategy for II:

$$t(A) = 5 \quad t(B) = 7$$

Player I has 18 possible strategies. Player II has 4 possible strategies.

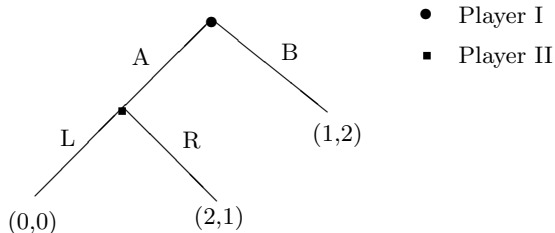
We have seen that if all the players in the game select a strategy and stick with it, then this totally determines the outcome of the game.

Let $s^* = (s_1^*, s_2^*)$ be a fixed strategy profile in a game. Then denote by $O(s^*)$ the resulting outcome. Note that the outcome may be a lottery.

Definition 2 A *Nash equilibrium* of an extensive form game with perfect information is a strategy profile $s^* = (s_1^*, s_2^*)$ such that

$$O(s^*) \succeq_i O(s_i, s_{-i}^*) \quad \forall i \in N.$$

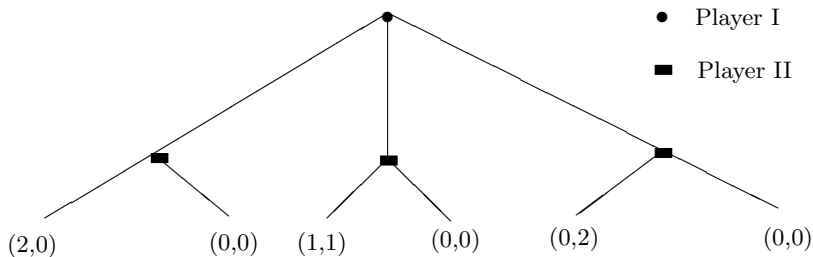
Example 4 Consider the following game in extensive form:



This game has two Nash equilibria: $(A; R)$ and $(B; L)$.

The equilibrium $(B; L)$ is supported by the “threat” of player 2 choosing L should player 1 choose A . This threat is not credible since player 2 has no way of committing himself to this choice.

Example 5 Two people use the following procedure to share two desirable identical indivisible objects. One of them proposes an allocation, which the other then either accepts or rejects. In the event of rejection neither person receives either of the objects. Each person cares only about the number of objects he obtains.



There are nine Nash equilibria, two of which do not rely on incredible threats.

Definition 3 A *subgame perfect equilibrium* of an extensive form game is a Nash equilibrium (s_1^*, s_2^*) that induces a Nash equilibrium at every subgame of the game.

To verify that a profile s^* is a subgame perfect equilibrium, the above definition requires us to check, for every player i and every subgame that there is no strategy that leads to an outcome that player i prefers.

The following result shows that in a game with finite horizon we can restrict attention to the first move of the player who makes it at every subgame.

Claim 1 The one deviation property In a finite horizon extensive form game with perfect information a strategy profile s^* is a subgame perfect equilibrium if and only if for each subgame the player who makes the first move cannot obtain a better outcome by changing only his initial action.

Proof: If s^* is a SPE of Γ , then it clearly satisfies the property. Suppose s^* is not a SPE. Then there is a subgame and a player who has a profitable deviation in it.

Take among all the profitable deviations of the player, the one that differs from the original strategy in the minimum number of nodes (there is no other profitable deviation that deviates in less nodes).

Consider the last node in which the deviating strategy and the original strategy differ and consider the subgame that starts there. It must be that in this subgame there is a profitable deviation which involves the first move of the player who moves. Otherwise, there must be a deviation in the original game with less deviations. \square

Theorem 1 Every finite game of perfect information has a subgame perfect equilibrium.

Proof: The proof is by induction on the length of the game-tree. It is clear that if the game-tree has length 1, there is a SPE: the agent who moves at the root of the tree has finitely many choices and therefore there is a best choice for him. Since the game has no proper subgames, the strategy that dictates that player I chooses his best move is a SPE.

Assume that every perfect information game of length at least k has a SPE and let Γ be a perfect information game of length $k + 1$. We want to show that Γ has a SPE. Consider the root of the game-tree and without loss of generality call the player that moves there Player I. At the root of the tree, Player I has finitely many actions: $A = \{a_1, \dots, a_n\}$. Each action leads to a node that is the root of a subtree. Denote the subgame that has action a as its root by $\Gamma(a)$. Note that $\Gamma(a)$ may be a tree with one node (a terminal node). By construction, for each $a \in A$, $\Gamma(a)$ is a game of length less or equal k , and by the induction hypothesis, it has a SPE. For each $a \in A$, fix a subgame perfect equilibrium of $\Gamma(a)$ and let $p(a)$ be player I's payoff at that subgame perfect equilibrium. We shall build a strategy profile that is a SPE of Γ . For this purpose we need to say what each player does at each node he is supposed to move. Consider a node that is not the root of the game-tree. Then it belongs to one and only one subgame $\Gamma(a)$. Let the player who moves at that node choose the same action that the subgame perfect equilibrium of $\Gamma(a)$ dictates him to choose. At the root of the tree, let player I choose an action that maximizes $p(a)$. That is he chooses $a \in A$ such that $p(a) \geq p(b)$ for all $b \in A$. In order to show that the strategy profile that we just defined is a SPE, we shall apply the one-deviation property. Let,

v be a node that is not the root of the tree. Then since v belongs to a subgame $\Gamma(a)$ and the players behave in $\Gamma(a)$ according to a SPE, the player who moves at v cannot profitably deviate at v in the subgame that starts at v . It remains to check that Player I cannot profitably deviate by choosing another action at the root. But by the way his action was chosen, there is no action in A that would lead to a higher payoff for Player I. Therefore, by the one deviation property, the strategy profile that we built is a SPE. \square