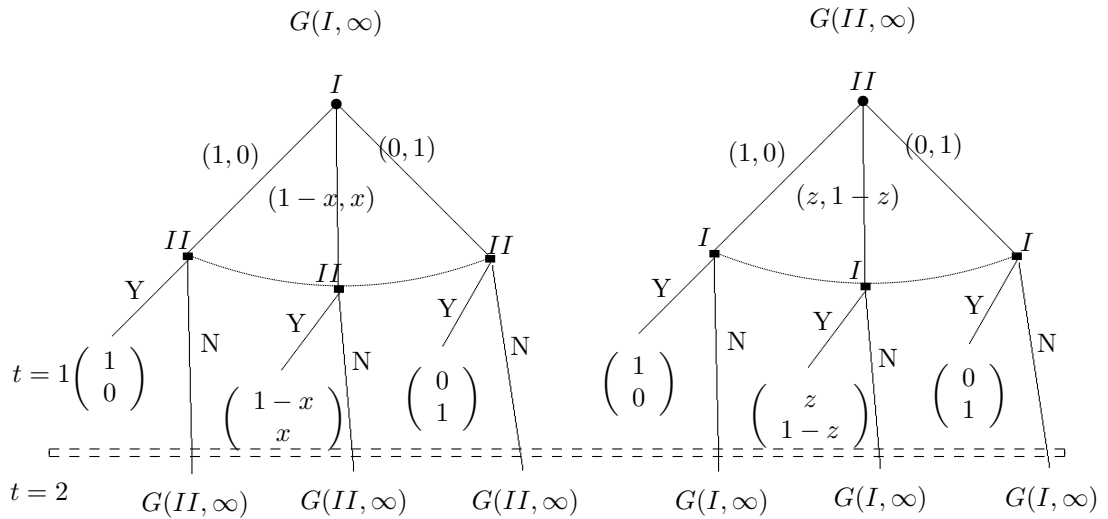


# 1 The Infinite Horizon Game

The description of the game, which is denoted  $G(I; \infty)$ , is as follows.

Player I makes a proposal  $(1 - x; x)$  in the first period, following which player II either accepts or rejects. If she accepts the proposal, the proposed division is implemented. If she rejects, we go into period 2 in which player II makes an offer. Player I then either accepts or rejects the offer. They continue in this way until a proposal is accepted. Note that nothing prevents all the proposals being refused, in which case the game will go on forever.

The game  $G(II; \infty)$  is similar to  $G(I; \infty)$  with the only difference that the first proposer is player II.



Again, the possible physical outcomes are the possible divisions of the pie together with the time period at which is carried out. They can be represented by an object of the form

$$\langle (1 - x, x); t \rangle \quad \text{where } t \in \mathbb{N} \quad \text{or} \quad \langle (0, 0); \infty \rangle.$$

The players evaluate the physical outcomes according to the following utility functions:

$$U_I(\langle(1-x, x); t\rangle) = (1-x)\delta_1^{t-1}$$

$$U_{II}(\langle(1-x, x); t\rangle) = x\delta_2^{t-1}$$

where  $\delta_1$  and  $\delta_2$  represent agent  $I$  and  $II$ 's discount factors, respectively.

**Lemma 1** The following pair of strategies is a SPE of  $G(I; \infty)$  and of  $G(II; \infty)$ :

Player I proposes

$$\left( \frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \right).$$

Player I accepts an offer  $(z, 1-z)$  if and only if

$$z \geq \delta_1 \frac{1-\delta_2}{1-\delta_1\delta_2}.$$

Player II proposes

$$\left( \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_2\delta_1} \right).$$

Player II accepts an offer  $(1-x, x)$  if and only if

$$x \geq \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}.$$

**Proof :** To show that this pair of strategies is a SPE we are going to apply the one deviation property. Namely, there cannot be any node at which the player who moves there finds a profitable deviation by changing his decision at that node. So consider a node where player I makes a proposal. According to the proposed pair of strategies he will get  $\frac{1-\delta_2}{1-\delta_1\delta_2}$  because his proposal will be accepted.

If he deviates and proposes more than  $\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$ , player II will accept and player I will consequently get less than  $\frac{1-\delta_2}{1-\delta_1\delta_2}$ . Therefore, proposing more than  $\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$  is not a profitable deviation.

If he deviates and proposes less than  $\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}$  player II will reject and the outcome will be

$$\left( \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_2\delta_1} \right)$$

but next period. Hence, the corresponding utility is

$$\frac{\delta_1^2(1-\delta_2)}{1-\delta_1\delta_2} \text{ which is less than } \frac{1-\delta_2}{1-\delta_1\delta_2}.$$

Therefore, there is no profitable deviation at a node where player I makes a proposal.

Consider now a node where player I responds to a proposal by player II.

Suppose the proposal is  $(z, 1-z)$  with

$$z \geq \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}.$$

Player I's strategy tells him to accept in which case he gets  $z$ . If he deviates and rejects, he will get  $\frac{1-\delta_2}{1-\delta_1\delta_2}$  but tomorrow, which is equivalent to  $\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$  today. Since  $z \geq \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$ , this is not a profitable deviation.

Suppose the proposal is  $(z, 1-z)$  with  $z < \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$ . According to his strategy he must reject in which case he will get  $\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$ . If he deviates and accepts, he gets only  $z$ .

An analogous reasoning shows that player II does not have a profitable deviation either. Therefore, the above pair of strategies constitutes a SPE.  $\square$

**Theorem 1** The infinite horizon bargaining game  $G(I; \infty)$  has a unique SPE outcome. The game  $G(II, \infty)$  has a unique SPE outcome too.

**Proof :** In principle, the game  $G(I; \infty)$  may have many SPE outcomes. The previous lemma shows that it has at least one. The same is true for the game  $G(II, \infty)$ .

Let  $A_1$  be the largest SPE payoff of player I in  $G(I, \infty)$  and let  $a_1$  be the smallest one. Analogously, let  $B_2$  be player II's largest SPE payoff in  $G(II, \infty)$  and let  $b_2$  be the player II's smallest SPE payoff.

Step 1: In  $G(I, \infty)$ , player II can always reject a proposal at time 1. In this case he will get at least  $b_2$  tomorrow which is equivalent to  $\delta_2 b_2$  today. Since a SPE has the one deviation properly, player II gets in  $G(I; \infty)$  at least  $\delta_2 b_2$  which means that player I gets at most  $1 - \delta_2 b_2$ , namely

$$A_1 \leq 1 - \delta_2 b_2. \quad (1)$$

Step 2: In  $G(II; \infty)$ , if player I refuses a proposal  $(z, 1 - z)$ , he will get at most  $A_1$  tomorrow, which is equivalent to  $\delta_1 A_1$  today. Therefore, player I must accept any offer  $(z, 1 - z)$  with  $z > \delta_1 A_1$ , in which case player II gets  $1 - z < 1 - \delta_1 A_1$ . This means that player II can guarantee any utility level lower but arbitrarily close to  $1 - \delta_1 A_1$ . Therefore, II gets in equilibrium at least  $1 - \delta_1 A_1$  or

$$b_2 \geq 1 - \delta_1 A_1. \quad (2)$$

By (1) and (2), we have

$$A_1 \leq 1 - \delta_2 b_2 \leq 1 - \delta_2(1 - \delta_1 A_1) = 1 - \delta_2 + \delta_1 \delta_2 A_1$$

or

$$A_1 \leq \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

Step 3: In  $G(II, \infty)$ , player I can always reject a proposal at time 1. In this case he will get at least  $a_1$  tomorrow which is equivalent to  $\delta_1 a_1$  today. Since a SPE has the one deviation properly, player I gets in  $G(II; \infty)$  at least  $\delta_1 a_1$  which means that player II gets at most  $1 - \delta_1 a_1$ , namely

$$B_2 \leq 1 - \delta_1 a_1. \quad (3)$$

Step 4: In  $G(I; \infty)$  if player II refuses a proposal  $(1 - x, x)$ , he will get at most  $B_2$  tomorrow, which is equivalent to  $\delta_2 B_2$  today. Therefore, player II

must accept any offer  $(1 - x, x)$  with  $x > \delta_2 B_2$ , in which case player I gets  $1 - x < 1 - \delta_2 B_2$ . This means that player II can guarantee any utility level lower but arbitrarily close to  $1 - \delta_2 B_2$ . Therefore, player I gets in equilibrium at least  $1 - \delta_2 B_2$  or

$$a_1 \geq 1 - \delta_2 B_2. \quad (4)$$

By (3) and (4), we have

$$a_1 \geq 1 - \delta_2 B_2 \geq 1 - \delta_2(1 - \delta_1 a_1) = 1 - \delta_2 + \delta_2 \delta_1 a_1$$

or

$$a_1 \geq \frac{1 - \delta_2}{1 - \delta_2 \delta_1}.$$

Therefore,

$$a_1 = A_1 = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}.$$

With a similar reasoning we get

$$b_2 = B_2 = \frac{1 - \delta_1}{1 - \delta_1 \delta_2}$$

completing the proof. □