Common knowledge of an aggregate of expectations
based on Nielsen, et al., Econometrica 1990, 1235-1239

1 The model

There is a finite set of states of the world, denoted by Ω. There is a set $N = \{1, 2, \ldots, n\}$, of individuals. Each individual is endowed with an information partition Π_i of the set Ω, and a common prior μ. We denote by $\Pi_i(\omega)$ the element of the partion Π_i that contains ω. Let $\mathcal{F}_i$ be the σ-algebra generated by Π_i.

Let $X : \Omega \to \mathbb{R}$ be a random variable. For each $i \in N$, let $X_i : \Omega \to \mathbb{R}$ be the expectation of X conditional on Π_i. Therefore,

$$\sum_{\omega \in A} X_i(\omega) = \sum_{\omega \in A} X(\omega)\mu(\omega) \qquad \text{for all } A \in \mathcal{F}_i. \tag{1}$$

Further, for any function $g : \mathbb{R} \to \mathbb{R}$, $g(X_i(\omega_1)) = g(X_i(\omega_2))$ for all $\omega_1, \omega_2 \in \Pi_i(\omega)$. As a result,

$$\sum_{\omega' \in \Pi_i(\omega)} g(X_i(\omega'))X_i(\omega')\mu(\omega') = \sum_{\omega' \in \Pi_i(\omega)} g(X_i(\omega'))X(\omega')\mu(\omega')$$

and more generally

$$\sum_{\omega \in A} g(X_i(\omega))X_i(\omega)\mu(\omega) = \sum_{\omega \in A} g(X_i(\omega))X(\omega)\mu(\omega) \qquad \text{for all } A \in \mathcal{F}_i. \tag{2}$$

Let $f_i : \mathbb{R} \to \mathbb{R}$ for each $i \in N$ be a strictly increasing function, and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x_1, \ldots, x_n) = \sum_{i \in N} f_i(x)$.

Fix $y \in \mathbb{R}$ and define the event

$$E = \{\omega \in \Omega : f(X_1(\omega), \ldots, X_n(\omega)) = y\}.$$

Theorem 1 Suppose E is common knowledge at ω^*. Then, the n conditional expectiations of X at ω^* are equal, that is, $X_1(\omega^*) = \cdots = X_n(\omega^*)$.

Proof : Since E is common knowledge at ω^*, there is an event $A \in \mathcal{F}_i$ for all $i \in N$, such that $\omega^* \in A \subseteq E$.

Let $X_0 = \frac{1}{\mu(A)} \sum_{\omega \in A} X(\omega)\mu(\omega)$ denote the conditional expectation of X given A. Therefore,

$$\begin{aligned} \sum_{\omega \in A} X_0\mu(\omega) &= \sum_{\omega \in A} X(\omega)\mu(\omega) \\ &= \sum_{\omega \in A} X_i(\omega)\mu(\omega) \end{aligned} \tag{3}$$

Since by assumption $f(X_1(\omega), \ldots, X_n(\omega))$ is constant on A, we have

$$0 = \sum_{\omega \in A} f(X_1(\omega), \ldots, X_n(\omega))(X(\omega) - X_0)\mu(\omega).$$

By definition of f,

$$\begin{aligned} 0 &= \sum_{\omega\in A}\sum_{i\in N} f_i(X_i(\omega))(X(\omega)-X_0)\mu(\omega) \\ &= \sum_{i\in N}\sum_{\omega\in A} f_i(X_i(\omega))(X(\omega)-X_0)\mu(\omega). \end{aligned}$$

Since $A \in \mathcal{F}_i$ for all $i \in N$, using (2) we get

$$0 = \sum_{i\in N}\sum_{\omega\in A} f_i(X_i(\omega))(X_i(\omega)-X_0)\mu(\omega). \tag{4}$$

Similarly, using (3), for all $i \in N$,

$$0 = \sum_{\omega\in A} f_i(X_0)(X_i(\omega)-X_0)\mu(\omega)$$

and hence

$$0 = \sum_{i\in N}\sum_{\omega\in A} f_i(X_0)(X_i(\omega)-X_0)\mu(\omega). \tag{5}$$

Combining equations (4) and (5) we get

$$0 = \sum_{i\in N}\sum_{\omega\in A}(f_i(X_i(\omega))-f_i(X_0))(X_i(\omega)-X_0)\mu(\omega).$$

Since f_i are strictly monotone, the above equality can be satisfied only if $X_i(\omega) = X_0$ for all $i \in N$ and all $\omega \in A$. $\square$

Can the requirement that f be additively separable be dispensed with? In other words, can we get a result similar to Theorem 1 assuming only that f is strictly increasing but not necessarily additively separable? The answer is negative, as the following example shows.

Example 1 Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ be the set of states of the world, each with probability $1/5$, and let $I = \{1, 2, 3, 4, 5, 6\}$ be the set of agents. The agents information partitions are given by

$$\begin{aligned} \Pi_1 &= \{\{\omega_1,\omega_3\},\{\omega_2,\omega_4,\omega_5\}\} \\ \Pi_2 &= \{\{\omega_1,\omega_4\},\{\omega_2,\omega_3,\omega_5\}\} \\ \Pi_3 &= \{\{\omega_1,\omega_5\},\{\omega_2,\omega_3,\omega_4\}\} \\ \Pi_4 &= \{\{\omega_2,\omega_3\},\{\omega_1,\omega_4,\omega_5\}\} \\ \Pi_5 &= \{\{\omega_2,\omega_4\},\{\omega_1,\omega_3,\omega_5\}\} \\ \Pi_6 &= \{\{\omega_2,\omega_5\},\{\omega_1,\omega_3,\omega_4\}\}. \end{aligned}$$

Consider the random variable $X : \Omega \to \mathbb{R}$ given by

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_1, \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

The conditional expectations, X_i of the agents are given by

	ω_1	ω_2	ω_3	ω_4	ω_5
X_1	1/2	1/3	1/2	1/3	1/3
X_2	1/2	1/3	1/3	1/2	1/3
X_3	1/2	1/3	1/3	1/3	1/2
X_4	1/3	1/2	1/2	1/3	1/3
X_5	1/3	1/2	1/3	1/2	1/3
X_6	1/3	1/2	1/3	1/3	1/2

Consider the function $f : \mathbb{R}^6 \to \mathbb{R}$ given by $f(x_{1,} \ldots, x_6) = 0$. This function is monotone in the domain of the possible five vectors of expectiations given by the above table. It is common knowledge that $f(X_1(\omega), \ldots, X_n(\omega)) = 0$. Still, the conditional expectations are not equal.