Unit Cost Function

Consider unit cost of good 1. Choose a_{L1} and a_{K1} to

$$\min c(w,r) = wa_{L1} + ra_{K1}$$

$$\text{Subject to: } 1 - F(a_{L1}, a_{K1}) = 0.$$

The Lagrangian function associated with this problem is:

$$\mathcal{L}(a_{L1}, a_{K1}, w, r, \lambda) = wa_{L1} + ra_{K1} + \lambda[1 - F(a_{L1}, a_{K1})]. \quad (1)$$

FOCs are:

$$w - \lambda F_L = 0,$$

$$r - \lambda F_K = 0,$$

$$1 - F(a_{L1}, a_{K1}) = 0.$$

It follows that $-\dfrac{da_{K1}}{da_{L1}} = \dfrac{w}{r} = \dfrac{F_L}{F_K},$ which is a well-known condition. The solutions are $a_{K1}(w,r)$ and $a_{L1}(w,r)$. Substituting these into the objective function, we get

$$c(w,r) = a_{L1}(w,r)w + a_{K1}(w,r)r.$$

Note that multiplying all factor prices by λ does not affect the wage-rent ratio, and hence does not affect a_{ij}'s. Thus,

$$c(\lambda w, \lambda r) = a_{L1}(w,r)\lambda w + a_{K1}(w,r)\lambda r = \lambda c(w,r).$$

That is, the unit cost function is HD (1).

Shephard's Lemma

$$\frac{\partial c}{\partial w} = a_{L1} + \left(\frac{w da_{L1} + r da_{K1}}{dw} \right) = a_{L1},$$

$$\frac{\partial c}{\partial r} = a_{K1} + \left(\frac{w da_{L1} + r da_{K1}}{dr} \right) = a_{K1},$$

which is also consistent with HD(1) of the unit cost function c(w,r),

$$c(w,r) = \frac{\partial c}{\partial w} w + \frac{\partial c}{\partial r} r = a_{L1}(w,r)w + a_{K1}(w,r)r. \qquad (2)$$

Then marginal functions, $a_{Lj}(w,r)$ and $a_{Kj}(w,r)$ are HD(0), i.e.,

$$\frac{\partial a_{Lj}}{\partial w} w + \frac{\partial a_{Lj}}{\partial r} r = 0,$$
$$\frac{\partial a_{Kj}}{\partial w} w + \frac{\partial a_{Kj}}{\partial r} r = 0, \qquad (3)$$

which holds if, and only if

$$a_{Lj}(w,r) = a_{Lj}(\lambda w, \lambda r),$$

$$a_{Kj}(w,r) = a_{Kj}(\lambda w, \lambda r),$$

Total Cost Approach

Usually, Shephard's lemma is expressed for total cost, rather than for unit cost.

Let $C(w,r) = wL + rK$.

Then Shephard's Lemma states:

$$\frac{\partial C}{\partial w} = L, \quad \frac{\partial C}{\partial r} = K.$$

Also, total cost is HD(1) in factor prices,

$$C(w,r) = \frac{\partial C}{\partial w} w + \frac{\partial C}{\partial r} r = wL + rK. \qquad (4)$$

Then the first derivatives, i.e., factor demand functions L(w,r) and K(w,r), are HD(0) in factor prices.

That is,

$$L(w,r)=\frac{\partial L}{\partial w}w+\frac{\partial L}{\partial r}r=0,\left(0\bullet L\right)$$
$$K(w,r)=\frac{\partial K}{\partial w}w+\frac{\partial K}{\partial r}r=0,\left(0\bullet K\right) \qquad (5)$$

which holds if, and only if $L(w,r)=L(\lambda w,\lambda r)$.

Cost minimization

- If we index labor demand by industries, for instance, $L_1(w,r)$ and $K_1(w,r)$.

Cost minimization implies wL_1+rK_1 is minimized, and hence $wdL_1+rdK_1=0$. Thus, for instance,

$$w\frac{\partial L_1(w,r)}{\partial w}+r\frac{\partial K_1(w,r)}{\partial w}=0.$$

Likewise,

$$w\frac{\partial L(w,r)}{\partial r}+r\frac{\partial K(w,r)}{\partial r}=0.$$

CONCAVITY OF COST FUNCTION $C(w,r)$

(This property was used in Choi "Factor growth and Equalized Factor Prices," IREF 2008.

- The indirect cost function, $c(w,r) = a_{L1}w + a_{K1}r$, obtained by minimizing the cost function which is linear in factor prices is now concave in factor prices.

- From any given point (w,r), a change in a factor price has a linear impact on the cost, if a_{ij} 's are constant, i.e., if no adjustment is made. (This is the case of the Leontief technology)

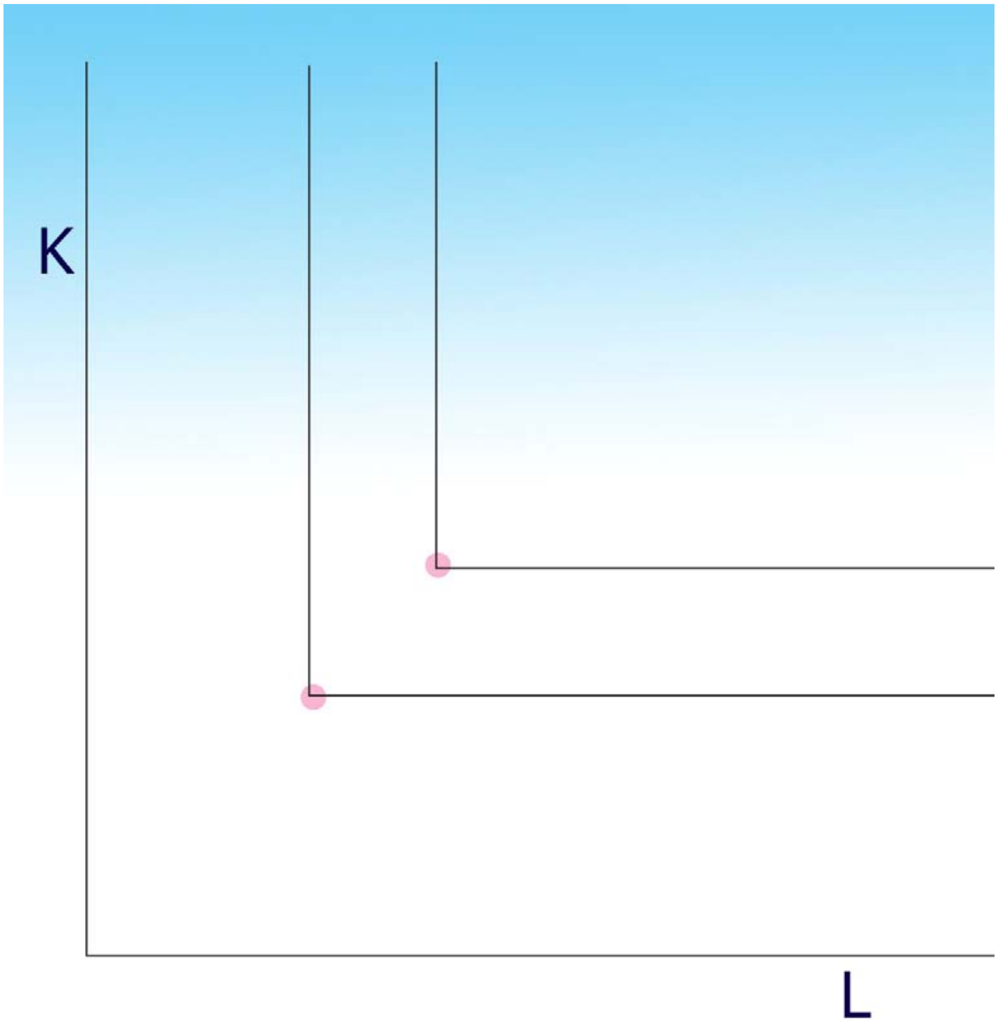
- If a_{ij} 's respond to a change in factor prices to minimize cost, then $c(w+\Delta w, r+\Delta r)$ will be below the cost with the Leontief technology, as illustrated in Figure 1.

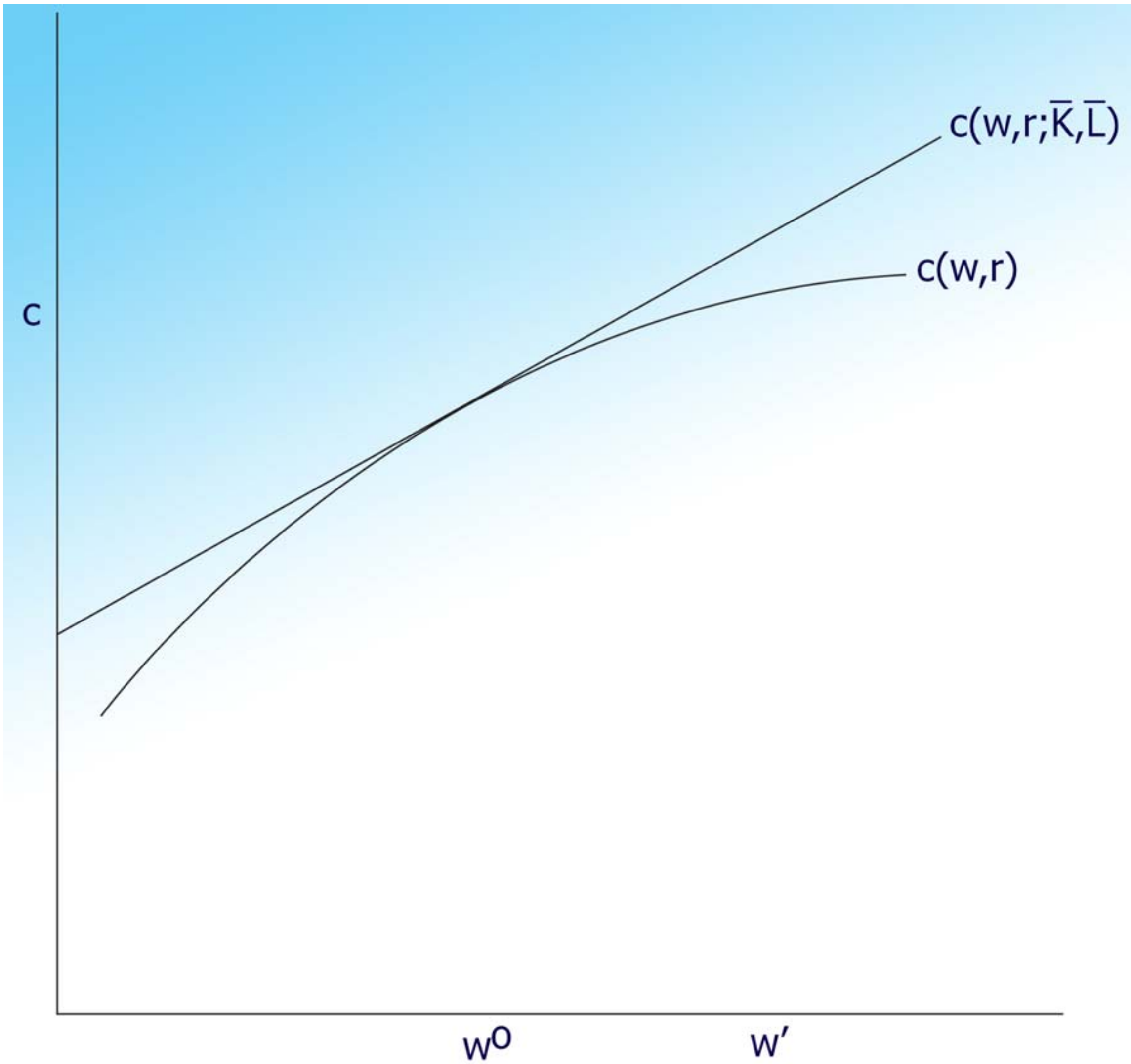
Leontief Technology

$$Y = \min_i (V_i / a_i) = \min(K/\alpha, L/\beta).$$

- Isoquants have kinks, not necessarily L-shaped. At every kink, there is no input substitution for a small change in factor prices.

Reason for concavity. With Leontief technology, C(w,r) is linear in factor prices. With all other production functions, a_{ij}'s respond to changes in factor prices and reduce the cost below what it would have cost under the Leontief technology.




c(w,r;$\bar{K}$,$\bar{L}$)
c(w,r)
c
w^{o}
w'

Properties of Short Run Cost Functions

One input, T (land) is fixed.

$$C(w,r,T) = wL + rK + sT$$

Then $C(w,r,T)$ is concave in (w,r) for the same reason.

Extra note

(1) Production function, $Y = F(K, L, T)$, is HD(1). $\Rightarrow$

$$F_K K + F_L L + F_T T = mY = Y. \qquad (6)$$

(2) If F(K,L) is HD(1), then Marginal Products are HD(0).

Differentiate (6) with respect to K,

$$F_{KK} K + F_{LK} L + F_{TK} T + F_K = F_K.$$

Or

$$F_{KK} K + F_{LK} L + F_{TK} T = 0 \bullet F_K = 0.$$

The same is true with MP_L.

Divide (6) by T. Average product of land is:

$$F_K (K/T) + F_L (L/T) + F_T = Y/T.$$

Is the APL homogenous of degree 1 in nonland inputs? No. In the two variable case, holding T constant, raising K increase APK first but lowers it later.