0. Preliminaries

Euler's Theorem and Constant Returns to Scale

Definition: f(x) is said to be homogenous of degree m [HD(m)], if and only if, for $\lambda > 0$,

$$f(\lambda x) = \lambda^m f(x), \text{ for all } x = (x_1, x_2, ..., x_n). \qquad (1)$$

Euler's Theorem: f(x) is homogeneous of degree m, if and only if

$$mf(x) = \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + ... + \frac{\partial f}{\partial x_n} x_n. \qquad (2)$$

Proof: Fix x and let $g(\lambda) = f(\lambda x)$. Differentiate the LHS of (1) with respect to λ,

$$g'(\lambda) = \frac{\partial f(\lambda x)}{\partial(\lambda x_1)} x_1 + \frac{\partial f(\lambda x)}{\partial(\lambda x_2)} x_2 + ...$$

When evaluated at $\lambda = 1$,

$$g'(1) = \frac{\partial f(x)}{\partial x_1} x_1 + \frac{\partial f(x)}{\partial x_2} x_2 + ... = \sum_{i=1}^{n} f_i x_i. \qquad (3)$$

Differentiating the RHS of (1), we get

$$g'(\lambda) = m\lambda^{m-1} f(x). \qquad (4)$$

When evaluated at λ = 1, $g'(1) = mf(x).$ Thus, we have Euler's Theorem.

Application to trade theory

Constant Returns to Scale Production Function = homogeneous of degree 1.

(1) Production function, $Y = F(K, L)$, is HD(1). ⇒

$$F_K K + F_L L = mY = Y. \qquad (5)$$

⇒ $p(F_K K + F_L L) = pY,$ or in competitive markets,

$wL + rK = pY.$ ⇒ Zero Profits always (even in the short run).

$\Rightarrow$ This is different from zero profits of competitive firms in Long Run Equilibrium.

(2) If F(K,L) is HD(1), then Marginal Products are HD(0).

Differentiate (5) with respect to K,

$$F_{KK}K + F_{LK}L + F_K = F_K.$$

Or

$$F_{KK}K + F_{LK}L = 0 \bullet F_K = 0.$$

The same is true with MP_L.

Similarly, if F(K,L,T) is HD(1), then

$$F_K K + F_L L + F_T T = Y. \qquad (6)$$

Differentiating (6) with respect to K gives

$$F_{KK}K + F_K + F_{LK}L + F_{TK}T = F_K.$$

Again, we get

$$F_{KK}K + F_{LK}L + F_{TK}T = 0.$$

● In general, even with many factors (n > 2), if the production function exhibits CRS, then marginal product of every input is HD(0).

Hat Calculus

Z = XY ⇔ $\hat{Z} = \hat{X} + \hat{Y}.$

Z = X/Y ⇔ $\hat{Z} = \hat{X} - \hat{Y}.$

Z = 1/Y ⇔ $\hat{Z} = -\hat{Y}.$

$Z = f(X, Y) \Leftrightarrow \varepsilon_X \hat{X} + \varepsilon_Y \hat{Y}.$ ($\varepsilon_X = \frac{\partial Z}{\partial X}\frac{X}{Z}$)

$Z = a_1 x_1 + a_2 x_2. \Leftrightarrow \hat{Z} = s_1 \hat{x}_1 + s_2 \hat{x}_2.$

($s_1 = \frac{a_1 x_1}{Z}$ = share of 1).

If ai's also change, then

$$\hat{Z} = s_1\left(\hat{a}_1 + \hat{x}_1\right) + s_2\left(\hat{a}_2 + \hat{x}_2\right).$$

$$Z = f\left(g(X),Y\right) \Leftrightarrow \varepsilon_g \varepsilon_X \hat{X} + \varepsilon_Y \hat{Y}.$$

$$\varepsilon_g = \frac{\partial Z}{\partial g}\frac{g}{Z}, \quad \varepsilon_X = \frac{\partial g}{\partial X}\frac{X}{g}$$

$$\Rightarrow \varepsilon_{Zg} \times \varepsilon_{gX} = \frac{\partial Z}{\partial g}\frac{g}{Z} \times \frac{\partial g}{\partial X}\frac{X}{g} = \frac{\partial Z}{\partial X}\frac{X}{g} = \varepsilon_{ZX}.$$