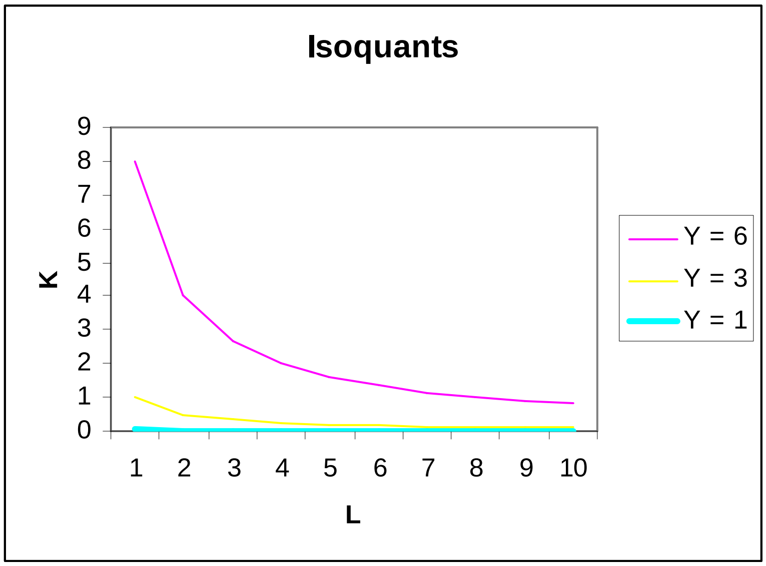
Example: Cobb-Douglas production function. f(L,K) = Y = 3 $K^{1/3}L^{1/3}$. Cost: Labor is \$9 and Captal is \$27.

$$Y = 3K^{1/3}L^{1/3} \Leftrightarrow$$

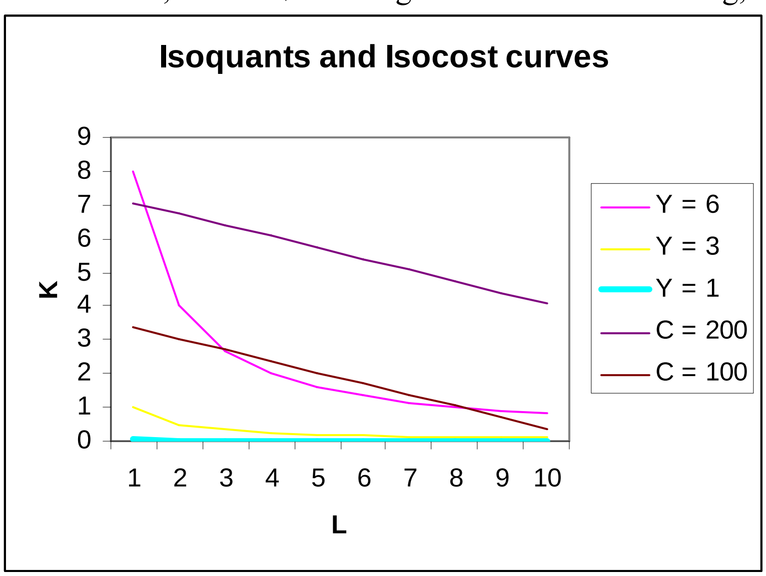
Isoquants: $$K^{1/3} = \frac{Y}{3L^{1/3}} \Leftrightarrow$$

$$K = \frac{Y^3}{27L}$$

To produce 3 units, we have K = 1/L. To produce 1 unit we have K=1/27L, to produce 6 units we have Y = 8/L.



Isocost curves: C(K,L) = 9L + 27K, so $K = \frac{C}{27} - \frac{1}{3}L$. So for \$200, we get K = 7.4 - ⅓L, and for \$100 we get K = 3.7 - ⅓L. Drawing, we get:



Recall that to minimize cost for a given quantity, we had to set MRTS=w_L/w_K. Suppose we're producing 6 units. What is the minimum cost?

$$MRTS = \frac{\partial f(L,K)/\partial L}{\partial f(L,K)/\partial K} = \frac{K^{1/3}L^{-2/3}}{K^{-2/3}L^{1/3}} = \frac{K}{L} \text{ so we need}$$

$$\frac{K}{L} = \frac{9}{27} \Leftrightarrow K = \frac{1}{3}L$$

What now? We wanted to produce 6 units, so $6 = 3K^{1/3}L^{1/3} = 3(\frac{1}{3}L)^{1/3}L^{1/3} = 3(\frac{1}{3})^{1/3}L^{2/3}$.

Solving for L, we get $L^{\frac{2}{3}} = \frac{2}{3^{\frac{1}{3}}} \Leftrightarrow L = \frac{2^{\frac{3}{2}}}{\frac{1}{3}^{\frac{1}{2}}} = 4.9$ and K = ⅓L = 1.6 and the total cost is

$$C = 9\frac{2^{\frac{3}{2}}}{\frac{1}{3}^{\frac{1}{2}}} + 27\frac{1}{3}\frac{2^{\frac{3}{2}}}{\frac{1}{3}^{\frac{1}{2}}} = 88.18$$

In general, suppose we want to find the cost as a function of the quantity produced.

$MRTS = \frac{K}{L}$ and the price ratio is ⅓, so $K=\frac{1}{3}L$ and $Q = 3K^{1/3}L^{1/3} = 3(\frac{1}{3}L)^{1/3}L^{1/3} = 3(\frac{1}{3})^{1/3}$

$L^{2/3}$. So $L = \left(\frac{Q}{3\left(\frac{1}{3}\right)^{\frac{1}{3}}}\right)^{\frac{3}{2}} = \left(\frac{Q}{3^{\frac{2}{3}}}\right)^{\frac{3}{2}} = \frac{Q^{\frac{3}{2}}}{3}$ and $K = \frac{Q^{\frac{3}{2}}}{9}$ So

$$C(Q) = 9\frac{Q^{\frac{3}{2}}}{3} + 27\frac{Q^{\frac{3}{2}}}{9} = 3Q^{\frac{3}{2}} + 3Q^{\frac{3}{2}} = 6Q^{\frac{3}{2}}$$

Long-run cost function.

Now, what happens if K is fixed at 1.6 and now we wish to produce 3 units? (SHORT RUN) Then we determine how much L we need to produce 3 when K is fixed at the previous level: 3 = 3

$$3 = 3K^{\frac{1}{3}}L^{\frac{1}{3}} = 3\left(\frac{2^{\frac{3}{2}}}{3\cdot\frac{1}{3}^{\frac{1}{2}}}\right)^{\frac{1}{3}}L^{\frac{1}{3}} \Leftrightarrow L^{\frac{1}{3}} = \frac{1}{\left(\frac{2^{\frac{3}{2}}}{3\cdot\frac{1}{3}^{\frac{1}{2}}}\right)^{\frac{1}{3}}} = \left(\frac{3\cdot\frac{1}{3}^{\frac{1}{2}}}{2^{\frac{3}{2}}}\right)^{\frac{1}{3}} \Leftrightarrow L = \frac{3^{\frac{1}{2}}}{2^{\frac{3}{2}}}$$

Then $C = 9\frac{3^{\frac{1}{2}}}{2^{\frac{3}{2}}} + 27\frac{2^{\frac{3}{2}}}{3 \cdot \frac{1}{3}^{\frac{1}{2}}} = 48.71$

In the long run we could have produced this at a cost of $C(3) = 6 \cdot 3^{\frac{3}{2}} = 31.18$

In general we can find the short run cost function like this: We fix K at κ. Then to produce Q units we have: $Q = 3\,\mathrm{k}^{\frac{1}{3}} L^{\frac{1}{3}} \Leftrightarrow L^{\frac{1}{3}} = \frac{Q}{3k^{\frac{1}{3}}} \Leftrightarrow L = \frac{Q^3}{27k}$ Then $C = 9\frac{Q^3}{27k} + 27k$.